

Analytical Bounds on the Average Error Probability for Nakagami Fading Channels

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Abstract—We investigate the average error probability of data communication over Nakagami fading channels. First, we discuss some new identities and properties of a certain integral representation of the average error probability. Second, we propose novel lower and upper bounds. Both bounds are sharp, and they have a simple closed-form representation. We also demonstrate that the bounds are very precise for a wide range of parameters. A relative error of less than 1.2% is achieved. Finally, the mathematical structure of the bounds is investigated. For both bounds, parameters can be adapted to achieve a simpler form, however, at the price of a reduced precision. The channel capacity for Nakagami fading is hard to determine in general. It is expected that by using the accurate bounds developed in this work precise approximations of the capacity can be achieved.

I. INTRODUCTION

Understanding the stochastic nature of fast-fading communication channels is essential to develop high data rate transceivers. In digital communication theory the bit-error rate of a signal disturbed by additive white Gaussian noise (AWGN) is well investigated. However more general and practically relevant fading channels are much less understood and investigated. The main reason is that analytical expressions become nearly intractable and require the use of complicated functions. Thus, numerical methods are typically applied in simulations to optimize, analyze or verify a communication system, or parts of it. These methods are indeed appropriate for many applications, but inadequate to truly understand and describe the behavior of transceivers and their performance. A prominent example is the explicit evaluation of the average error probability (AEP) in signal transmission over Nakagami- or Rice-distributed fading channels. It is mathematically challenging to derive a closed-form equation of the AEP from its integral representation, if at all possible. Closed-form and analytical solutions are only known for some special cases. In this situation, mathematical approximations by simpler functions are of great help and fully sufficient for most practical purposes. The main topic of the present paper is hence to provide such approximations in the form of analytical bounds, and at the same time guarantee the minimum deviation from the true values.

In [1], we have provided a bunch of mathematical tools for dealing with the complicated AEP in Nakagami-distributed fading channels. The main focus of the present work is to continue the investigation of the AEP and provide novel bounds for the so called Beta-Nakagami integral (BeNaI).

First, we represent some new identities, proposed recently in [1], and based on them we devise a proper class of functions for bounding the BeNaI. Second, we propose analytical bounds on the BeNaI and discuss their mathematical properties. Our aim is to suggest a lower and an upper bound which are mathematically simple and accurate over a wide range of parameters. We are analytically able to show that the maximum relative error between the proposed bounds and the true BeNaI is less than 1.2%. Finally, we compute selected results numerically in order to visualize and demonstrate the achieved accuracy.

To the best of our knowledge, the investigation of the BeNaI and the corresponding bounds in this paper are new. They have been examined neither in the original work by Nakagami [2] nor in other publications. The main reason is that approximating the integrand of the BeNaI by simpler functions is much easier than the approximation of the integral itself. Thus, in many publications we can find approximations for the Gauss error function or for the Bessel function which are subsequently used to approximate the AEP of a Nakagami- or Rice-distributed data transmission, respectively. For instance, in [3]–[10] and [11] relatively good approximations of the integrands are suggested. In general, there exists a trade-off between improvement of integrand approximation and complication of the subsequent integration. Some examples of sharp integrand approximations, however not including the integration of the conditional error probability, are given in [12]–[14] and [15]. We also want to mention some pioneering works, like [16]–[21] and [22], which include the error probability and some corresponding approximations in terms of special functions and finite or infinite series. These approximations are rather useful for numerical evaluations. The approximation of the whole integral by lower and upper bounds has been an open problem to date, and is the main objective of the present paper.

In the next section, we start with prerequisite mathematical notations and definitions. Thereafter, the BeNaI is represented in its integral forms. Subsequently, we use particular properties of the BeNaI and, as a main result, we present sharp analytical upper and lower bounds for the BeNaI. Afterwards, some important mathematical properties of the bounds are discussed. In between, selected results are visualized by the corresponding curves. For the sake of compactness, some proofs of the central theorems are moved into [23].

II. MATHEMATICAL PREPARATION

Throughout this paper we use the same notation as given in [1] and [23]. The sets of positive (non-negative) integers and (non-negative) real numbers are denoted by \mathbb{N} (\mathbb{N}_0) and $(\mathbb{R}_+) \mathbb{R}$, respectively. By $|z|$ we denote the absolute value of some real number z . A function $\vartheta(x)$ is said to be of order $\mathcal{O}(\omega(x))$ as $x \mapsto x_0$ whenever for some $\delta > 0$, $\epsilon > 0$ and all x with $|x - x_0| < \epsilon$ the inequality $|\vartheta(x)| \leq \delta |\omega(x)|$ holds. We further introduce some important special functions and summarize their properties, which may be found for example in chapters 6, 7, 15 and 26 of [24].

Euler's classical gamma function is defined as

$$\Gamma(x) := \int_0^{\infty} t^{x-1} e^{-t} dt, \quad x > 0. \quad (1)$$

It is well-known that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma(1) = \Gamma(2) = 1 \quad \text{and} \quad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}. \quad (2)$$

Moreover, for all positive real numbers x , the identity

$$\Gamma(x+n) = \Gamma(x) \prod_{i=0}^{n-1} (x+i), \quad n \in \mathbb{N}, \quad (3)$$

holds, which especially entails $\Gamma(x+1) = x\Gamma(x)$. Furthermore, it holds that

$$\lim_{n \rightarrow \infty} n^{b-a} \frac{\Gamma(n+a)}{\Gamma(n+b)} = 1, \quad a, b \in \mathbb{R}_+. \quad (4)$$

We will also use the digamma function

$$\psi(x) := \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x > 0, \quad (5)$$

where $\Gamma'(x)$ denotes the first derivative of $\Gamma(x)$.

Closely related to the gamma function is the incomplete beta function, which is defined as

$$B(a, b; x) := \int_0^x t^{a-1} (1-t)^{b-1} dt, \quad (6)$$

for all $a, b > 0$ and $0 \leq x \leq 1$. By substituting t by $\frac{t}{1+t}$ in (6), we obtain the identity

$$B(a, b; x) = \int_0^{\frac{x}{1-x}} \frac{t^{a-1}}{(1+t)^{a+b}} dt. \quad (7)$$

The beta function is obtained as $x \rightarrow 1$ yielding

$$B(a, b) := \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = B(b, a) = \int_0^{\infty} \frac{t^{a-1}}{(1+t)^{a+b}} dt. \quad (8)$$

The Gaussian error function and a useful series expansion are given for $x \in \mathbb{R}$ by

$$\operatorname{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt = 1 - \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}. \quad (9)$$

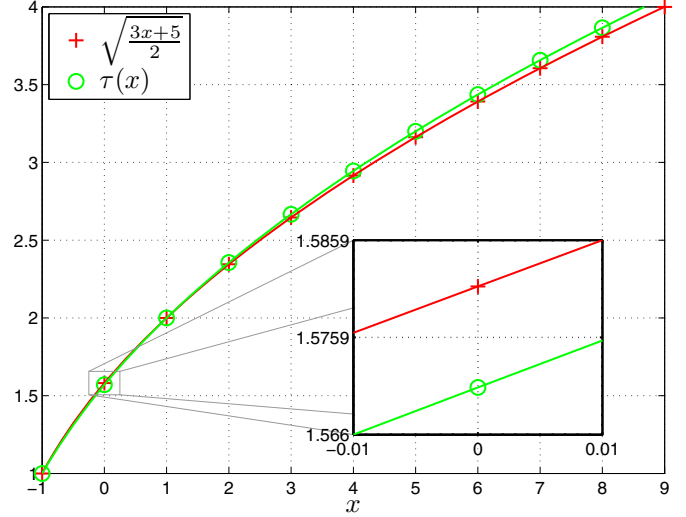


Fig. 1. The ratio $\tau(x)$ and its associated bounds from Theorem II.2 are visualized for the range of $-1 \leq x \leq 9$. For all $x \geq 1$ the inequality $\sqrt{(3x+5)/2} \leq \tau(x)$ holds while for all $-1 \leq x \leq 1$ the reverse inequality holds. The curves are closely adjacent to one another such that the maximum relative error is less than $\sqrt{\pi/3} - 1$ or equivalently 2.33%. The relative error approaches its maximum as x approaches infinity. The curves intersect one another at $x = -1$ and at $x = 1$.

Another special function of importance for this work is the Gaussian hypergeometric function. For all a, b and $c \in \mathbb{R}_+$ and for all $x \in \mathbb{R}$, it is defined as

$${}_2F_1(a, b; c; x) := \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)x^n}{\Gamma(c+n)\Gamma(n+1)}. \quad (10)$$

If $a, b > 0$ and $c > b$, then for all $x > 0$ the integral representation

$${}_2F_1(a, b; c; -x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1+xt)^a} dt \quad (11)$$

holds.

Definition II.1 For all real numbers $x > -2$, we define the ratio

$$\tau(x) := \frac{\sqrt{\pi} \Gamma\left(\frac{x+3}{2}\right)}{\Gamma\left(\frac{x+2}{2}\right)}. \quad (12)$$

Obviously, $\tau(x)$ is positive. Furthermore, the identities

$$\tau(-1) = 1, \quad \tau(0) = \frac{\pi}{2} \quad \text{and} \quad \tau(1) = 2 \quad (13)$$

easily follow from (2). From (3) and (8) we conclude that

$$\tau(x) = \frac{x+1}{2} \int_0^{\infty} \frac{dt}{(1+t)^{\frac{x+2}{2}} \sqrt{t}}. \quad (14)$$

The ratio $\tau(x)$ in (12) and its properties have been investigated in a series of papers. W. Gautschi [25] proposed a lower and an upper bound for this ratio, which are together known as Gautschi's double inequality. This double inequality was

later refined and extended by many scientists, primarily by D. Kershaw [26], M. Merkle [27] and F. Qi [28]. For the purpose of the present paper, Gautschi's double inequality plays an important role. However, Gautschi's bounds and subsequent improvements are too weak, particularly for the case $-1 \leq x < 1$, in order to prove some of the main theorems of the present work. Thus, a new tighter inequality is given in the following theorem.

Theorem II.2 *For all real numbers $x \geq 1$, the inequality*

$$\sqrt{\frac{3x+5}{2}} \leq \tau(x) \quad (15)$$

holds. If $-1 \leq x \leq 1$, then the reverse inequality holds.

Proof: See [23, pp. 9-14]. ■

In Figure 1, both sides of inequality (15) are depicted. As can be seen from the graph, both lower and upper bounds are very close to $\tau(x)$ in their respective range and equality holds for $x = -1$ and $x = 1$.

III. DESCRIPTION OF THE PROBLEM AND ASSOCIATED IDENTITIES

The AEP for communication over Nakagami-distributed fading channels, assuming a coherent transmission, is usually given by its integral form [1, eq. 15]. Simple substitutions and normalizations yield the following concise form.

Definition III.1 *For all $p \in \mathbb{R}$, $-1 < p < \infty^1$ and for all $x \in \mathbb{R}_+$, we have given the name Beta-Nakagami Integral (BeNaI) to*

$$g(p, x) := \frac{(p+1)\sqrt{\pi}}{\Gamma(\frac{p+2}{2})} \int_0^\infty t^p e^{-xt^2} \operatorname{erfc}(t) dt. \quad (16)$$

In [1, Theorem 6, eq. 27b], we have shown, among other properties and identities, the convenient identity

$$g(p, x) = \frac{p+1}{2} \int_0^1 \frac{t^{\frac{p-1}{2}}}{(1+xt)^{\frac{p+2}{2}}} dt, \quad (17)$$

which enables us to evaluate the value of $g(p, x)$, for certain p and x , numerically more accurate than the evaluation of $g(p, x)$ by equation (16). There are three reasons for this fact. First, the integration domain is finite and it only includes a singularity at the origin for all $-1 < p < 1$. Second, the integrand does not contain any complicated functions and is merely an integration over a broken rational function. Third, all operations and functions in the integrand are numerically stable. Furthermore, we have shown that by using the new

identity in (17), one can deduce some closed-form solutions of (16) easily, e.g.,

$$g(1, x) = \frac{2}{1+x+\sqrt{1+x}}, \quad (18a)$$

$$g(0, x) = \frac{1}{\sqrt{x}} \arctan(\sqrt{x}) \quad (18b)$$

and

$$\lim_{p \rightarrow -1} g(p, x) = 1. \quad (18c)$$

Moreover, the derivation of some specific series expansions² becomes more comfortable as shown in [1, eq. 40 and eq. 42], e.g.,

$$g(p, x) = 1 - \frac{(p+1)(p+2)x}{2(p+3)} + \frac{(p+1)(p+2)(p+4)x^2}{8(p+5)} - \mathcal{O}(x^3) \quad (19a)$$

and

$$g(p, x) = \frac{\tau(p)}{x^{\frac{p+1}{2}}} - \frac{(p+1)}{x^{\frac{p+2}{2}}} + \frac{(p+1)(p+2)}{6x^{\frac{p+4}{2}}} - \mathcal{O}(x^{-\frac{p+6}{2}}). \quad (19b)$$

However, the concise form in (17) has closed-form solutions only for some particular cases, and is thus poorly applicable for analytical treatments. Hence, we set out to find analytical bounds for $g(p, x)$, which are more applicable in theoretical investigations. We start with the following experiments to find a parametric class of functions for an accurate determination of proper bounds.

Since $\sqrt{1+xt}$ is greater than or equal to one and \sqrt{t} is less than or equal to one, we can divide the integrand in (17) by these quantities in order to obtain a lower and an upper bound, respectively. Hence, it holds the double inequality

$$\frac{p+1}{2} \int_0^1 \frac{t^{\frac{p-1}{2}} dt}{(1+xt)^{\frac{p+3}{2}}} \leq g(p, x) \leq \frac{p+1}{2} \int_0^1 \frac{t^{\frac{p-2}{2}} dt}{(1+xt)^{\frac{p+2}{2}}}. \quad (20)$$

By using the identity³

$$\int \frac{t^{\rho-1} dt}{(1+xt)^{\rho+1}} = \frac{t^\rho}{\rho(1+xt)^\rho}, \quad \rho > 0, \quad (21)$$

we can calculate both sides of the double inequality (20) in closed-form and conclude

$$\frac{1}{(1+x)^{\frac{p}{2}}} \cdot \frac{1}{\sqrt{1+x}} \leq g(p, x) \leq \frac{1}{(1+x)^{\frac{p}{2}}} \cdot \frac{p+1}{p}. \quad (22)$$

As can be seen, a proper approximation is described by a product of two functions. The first function is the mutual factor on both sides of the above double inequality which may be considered as the main part of a proper approximation. The second function is a function which should lie between $\frac{1}{\sqrt{1+x}}$ and $\frac{p+1}{p}$ on the one hand, and on the other hand, the entire

¹The domain of p may be extended to $\{p \in \mathbb{R} \mid p > -1\} \cup \{\infty\}$, and therewith, all results can be proven by concepts of *uniform integrability*. But this extension is pointless for the purpose of the present work and hence is not considered.

²It is sometimes necessary to deal with another representation of the asymptotic expansion than that given in (19b). In such cases, we refer the reader to the book [29].

³Identity (21) is devised in the present work. However, we think that this identity is already well-known.

product of both functions should have the same properties of the BeNaI as discussed in [1, Section IV]. As we will see later, for the second function the choice of the form

$$\frac{e_1}{e_2 + \sqrt{1+x}e_3}, \quad e_1, e_2, e_3 \in \mathbb{R}_+, \quad (23)$$

is accurate enough for most applications.

IV. BOUNDS FOR THE BENAI

In this section, we propose two new bounds for the BeNaI from Definition III.1. We will determine bounds in a parametric class of functions given by

$$\frac{1}{(1+x)^{\frac{p}{2}}} \frac{e_1}{e_2 + \sqrt{1+x}e_3} \quad (24)$$

with positive real coefficients e_1 , e_2 and e_3 . This specific class of functions has advantageous properties to bound the BeNaI as we will describe later. Since the BeNaI depends on x and p , the coefficients e_1 , e_2 and e_3 must also depend on p to achieve accurate bounds. In the following, we first present both bounds with optimal coefficients. The particular choice of the coefficients will be explained later. Second, we introduce an important property of the bounds with respect to their coefficients, which enables the users to choose other coefficients in order to adapt the bounds for their needs.

Theorem IV.1 *Let $f(p, x)$ be defined by*

$$f(p, x) := \frac{1}{(1+x)^{\frac{p}{2}}} \frac{a_f(p) + 1}{a_f(p) + \sqrt{1+x}b_f(p)} \quad (25)$$

with the coefficients

$$a_f(p) := \frac{2}{p+3} \tau^2(p) - 1 \quad (26a)$$

and

$$b_f(p) := \frac{4 \tau^2(p)}{(p+3)^2}. \quad (26b)$$

Then for all $p \in \mathbb{R}$, $1 \leq p < \infty$ and for all $x \in \mathbb{R}_+$, the inequality

$$f(p, x) \leq g(p, x) \quad (27)$$

holds. If $p \in \mathbb{R}$ and $-1 < p \leq 1$, then the reverse inequality holds.

Proof: See [23, pp. 113-114]. ■

Remark IV.2 *The coefficients in (24) with $e_1 = a_f(p) + 1$, $e_2 = a_f(p)$ and $e_3 = b_f(p)$ are the best possible ones for the inequality (27) to hold. In other words, no coefficient can be replaced by a better value while keeping the other ones fixed in order to further improve the bound. In this sense, the inequality in Theorem IV.1 is sharp.*

Please note that by incorporating (26a) into (26b) we obtain the relationship

$$b_f(p) = 2 \frac{a_f(p) + 1}{p+3}. \quad (28)$$

We now collect some important properties of the coefficients $a_f(p)$ and $b_f(p)$.

Lemma IV.3 *For all $-1 < p < \infty$,*

- the coefficient $a_f(p)$ is strictly increasing in p ,*
- it holds that $0 < a_f(p) < \pi - 1$, and*
- both coefficients $a_f(p)$ and $b_f(p)$ are non-negative.*

Proof: The coefficient $a_f(p)$ is strictly increasing, if its first derivative with respect to p is positive. The first derivative is given by

$$\frac{da_f(p)}{dp} = 2\tau(p) \frac{2(p+3)\tau'(p) - \tau(p)}{(p+3)^2} \quad (29)$$

where $\tau'(p)$ denotes the first derivative of $\tau(p)$ with respect to p . The positivity is given, if the inequality

$$\frac{\tau'(p)}{\tau(p)} > \frac{1}{2(p+3)} \Leftrightarrow \psi\left(\frac{p+3}{2}\right) - \psi\left(\frac{p+2}{2}\right) > \frac{1}{p+3} \quad (30)$$

holds. After replacing p with $2(x-1)$, we can use [28, Theorem 3] to deduce

$$\psi\left(x + \frac{1}{2}\right) - \psi(x) > \frac{1}{2x+1} \quad (31)$$

which proves the statement.

Considering the monotonicity of $a_f(p)$, as shown above, we obtain the lower bound for $p \mapsto -1$ and the upper bound for $p \mapsto \infty$. For $p \mapsto -1$, we obtain from (13) the equality $\tau(-1) = 1$ and hence

$$a_f(-1) = \frac{2}{-1+3} \tau^2(-1) - 1 = 0. \quad (32)$$

By using the limit in (4) and replacing p with $2(x-1)$, we obtain

$$\lim_{p \rightarrow \infty} a_f(p) = -1 + \pi \lim_{x \rightarrow \infty} \left(\frac{\Gamma(x + \frac{1}{2})}{\sqrt{x}\Gamma(x)} \right)^2 = -1 + \pi. \quad (33)$$

The coefficient $b_f(p)$ is trivially non-negative by definition. Due to $0 < a_f(p) < \pi - 1$, the coefficient $a_f(p)$ is also non-negative. ■

Corollary IV.4 *By (28) the representation*

$$f(p, x) = \frac{1}{(1+x)^{\frac{p}{2}}} \frac{a_f(p) + 1}{a_f(p) + \sqrt{1+2x} \frac{a_f(p)+1}{p+3}} \quad (34)$$

is obtained, which is a strictly increasing function in $a_f(p)$.

Proof: We show that the first partial derivative of $f(p, x)$ with respect to $a_f(p)$ is strictly positive for all $-1 < p < \infty$. The derivative is given by

$$\begin{aligned} & (1+x)^{\frac{p}{2}} \frac{\partial f(p, x)}{\partial a_f(p)} \\ &= \frac{1+x \frac{a_f(p)+1}{p+3} - \sqrt{1+2x} \frac{a_f(p)+1}{p+3}}{\left[a_f(p) + \sqrt{1+2x} \frac{a_f(p)+1}{p+3} \right]^2 \sqrt{1+2x} \frac{a_f(p)+1}{p+3}}. \quad (35) \end{aligned}$$

Elementary algebra shows that the numerator of (35) is strictly positive. ■

Theorem IV.5 Let $h(p, x)$ be defined by

$$h(p, x) := \frac{1}{(1+x)^{\frac{p}{2}}} \frac{a_h(p) + 1}{a_h(p) + \sqrt{1+x} b_h(p)} \quad (36)$$

with the coefficients

$$a_h(p) := \frac{(p+1)}{\tau^2(p) - (p+1)} \quad (37a)$$

and

$$b_h(p) := \left[\frac{\tau(p)}{\tau^2(p) - (p+1)} \right]^2. \quad (37b)$$

Then for all $p \in \mathbb{R}$, $1 \leq p < \infty$ and for all $x \in \mathbb{R}_+$, the inequality

$$g(p, x) \leq h(p, x) \quad (38)$$

holds. If $p \in \mathbb{R}$ and $-1 < p \leq 1$, then the reverse inequality holds.

Proof: See [23, pp. 114-116]. ■

Remark IV.6 The coefficients in (24) with $e_1 = a_h(p) + 1$, $e_2 = a_h(p)$ and $e_3 = b_h(p)$ are the best possible ones for inequality (38) to hold. In other words, no coefficient can be replaced by a better value while keeping the other ones fixed in order to further improve the bound. In this sense, the inequality in Theorem IV.5 is sharp.

Please note that the bounds in Theorem IV.1 and IV.5 are converse to each other. For all $1 \leq p < \infty$ the double inequality $f(p, x) \leq g(p, x) \leq h(p, x)$ holds while for all $-1 < p \leq 1$ the converse double inequality $h(p, x) \leq g(p, x) \leq f(p, x)$ holds.

Note that by incorporating (37a) into (37b) we obtain the relationship

$$b_h(p) = \frac{a_h(p) + 1}{p+1} a_h(p). \quad (39)$$

Analogously to the properties of $a_f(p)$ and $b_f(p)$ the following holds.

Lemma IV.7 For all $-1 < p < \infty$,

- the coefficient $a_h(p)$ is strictly increasing in p ,
- it holds that $0 < a_h(p) < \frac{2}{\pi-2}$, and
- both coefficients $a_h(p)$ and $b_h(p)$ are non-negative.

Proof: The coefficient $a_h(p)$ is strictly increasing, if its first derivative with respect to p is positive. The first derivative is given by

$$\frac{da_h(p)}{dp} = \tau(p) \frac{\tau(p) - 2(p+1)\tau'(p)}{[\tau^2(p) - (p+1)]^2}. \quad (40)$$

The positivity is given, if the inequality

$$\frac{\tau'(p)}{\tau(p)} < \frac{1}{2(p+1)} \Leftrightarrow \psi\left(\frac{p+3}{2}\right) - \psi\left(\frac{p+2}{2}\right) < \frac{1}{p+1} \quad (41)$$

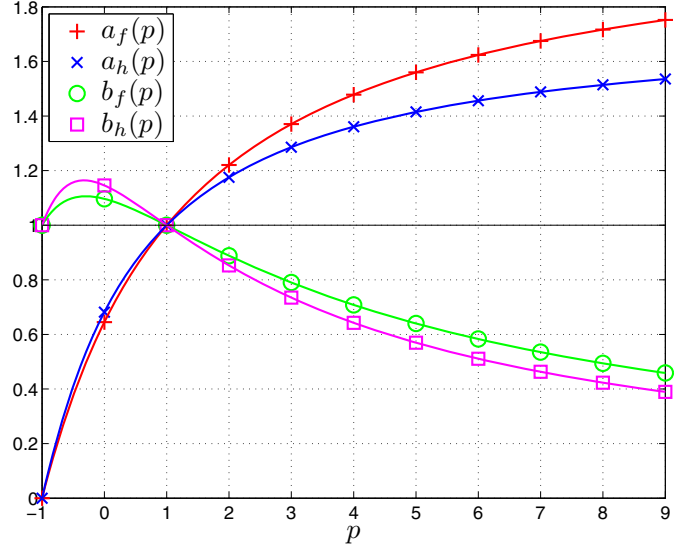


Fig. 2. The coefficients $a_f(p)$, $b_f(p)$, $a_h(p)$ and $b_h(p)$ are visualized for the range of $-1 < p \leq 9$. $a_f(p)$ and $a_h(p)$ are strictly increasing and they tend toward $\pi - 1$ and $2/(\pi - 2)$ as p approaches infinity, respectively. The coefficients $b_f(p)$ and $b_h(p)$ approach zero as p approaches infinity. Moreover, all coefficients are non-negative for all $-1 < p < \infty$.

holds. After replacing p with $2(x-1)$ we can again use [28, Theorem 3] to deduce

$$\psi\left(x + \frac{1}{2}\right) - \psi(x) < \frac{1}{2x-1} \quad (42)$$

which proves the statement.

Considering the monotonicity of $a_h(p)$, as shown above, we obtain the lower bound for $p \mapsto -1$ and the upper bound for $p \mapsto \infty$. For $p \mapsto -1$, we obtain from (13) the equality $\tau(-1) = 1$ and hence

$$a_h(-1) = \frac{(-1+1)}{\tau^2(-1) - (-1+1)} = 0. \quad (43)$$

By using the limit in (4) and replacing p with $2(x-1)$, we obtain

$$\begin{aligned} \lim_{p \rightarrow \infty} a_h(p) &= \lim_{p \rightarrow \infty} \frac{1}{\frac{\tau^2(p)}{p+1} - 1} \\ &= \lim_{x \rightarrow \infty} \frac{1}{\frac{\pi}{2} \left(\frac{\Gamma(x+\frac{1}{2})}{\sqrt{x}\Gamma(x)} \right)^2 - 1} = \frac{1}{\frac{\pi}{2} - 1}. \end{aligned} \quad (44)$$

The coefficient $b_h(p)$ is trivially non-negative by definition. Due to $0 < a_h(p) < \frac{2}{\pi-2}$, the coefficient $a_h(p)$ is also non-negative. ■

In Figure 2, the coefficients $a_f(p)$, $b_f(p)$, $a_h(p)$ and $b_h(p)$ are depicted. We can observe additional properties of these coefficients which are not important for our purpose and are thus not discussed further.

Corollary IV.8 By (39) the representation

$$h(p, x) = \frac{1}{(1+x)^{\frac{p}{2}}} \frac{a_h(p) + 1}{a_h(p) + \sqrt{1+x} \frac{a_h(p)+1}{p+1} a_h(p)} \quad (45)$$

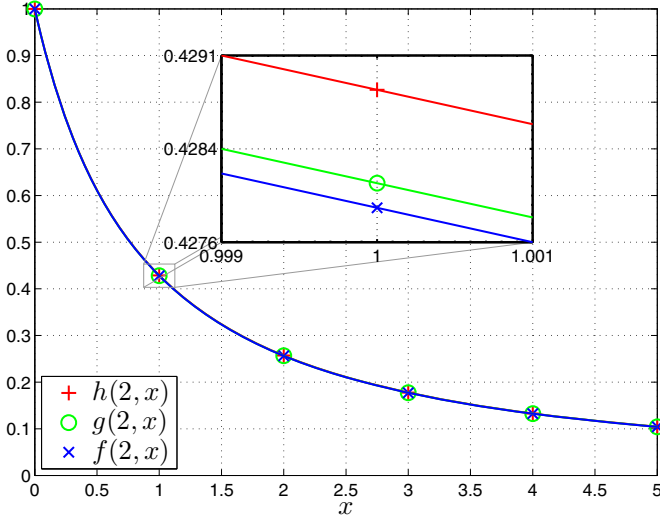


Fig. 3. The double inequality $f(p, x) \leq g(p, x) \leq h(p, x)$ is visualized for the range of $0 \leq x \leq 5$ and in the case of $p = 2$. The curves are closely adjacent to one another such that without magnification the differences are not really visible. The curves are equal at $x = 0$ and they approach zero as x approaches infinity.

is obtained, which is a strictly decreasing function in $a_h(p)$.

Proof: We show that the first partial derivative of $h(p, x)$ with respect to $a_h(p)$ is strictly negative for all $-1 < p < \infty$. The derivative is given by

$$\begin{aligned} & 2(1+x)^{\frac{p}{2}} \sqrt{1+x \frac{a_h(p)+1}{p+1} a_h(p)} \cdot \frac{\partial h(p, x)}{\partial a_h(p)} \\ &= \frac{2-x \frac{a_h(p)+1}{p+1} - 2\sqrt{1+x \frac{a_h(p)+1}{p+1} a_h(p)}}{\left[a_h(p) + \sqrt{1+x \frac{a_h(p)+1}{p+1} a_h(p)} \right]^2}. \quad (46) \end{aligned}$$

By simple rearrangement of the numerator, we derive the assertion from the chain of inequalities

$$\begin{aligned} & x \frac{a_h(p)+1}{p+1} + 2\sqrt{1+x \frac{a_h(p)+1}{p+1} a_h(p)} \\ & \geq 0 + 2\sqrt{1+x \frac{a_h(p)+1}{p+1} a_h(p)} \geq 2\sqrt{1+0}. \quad (47) \end{aligned}$$

■ and

Remark IV.9 Because of Corollary IV.4, $f(p, x)$ is monotonically increasing in $a_f(p)$. The coefficient $a_f(p)$, in turn, is increasing in $\tau(p)$ due to the relationship (26a). Choosing an upper bound of $\tau(p)$, which is desirably easier to handle than $\tau(p)$ itself, also yields an upper bound for $f(p, x)$. By this, we also obtain a weaker upper bound of $g(p, x)$ for $-1 < p \leq 1$ of potentially simpler form. Analogously, selecting a lower bound of $\tau(p)$ yields a weaker lower bound of $g(p, x)$ for $1 \leq p < \infty$.

Analogously, because of Corollary IV.8 and relationship (37a) any surrogate function, which is greater or less

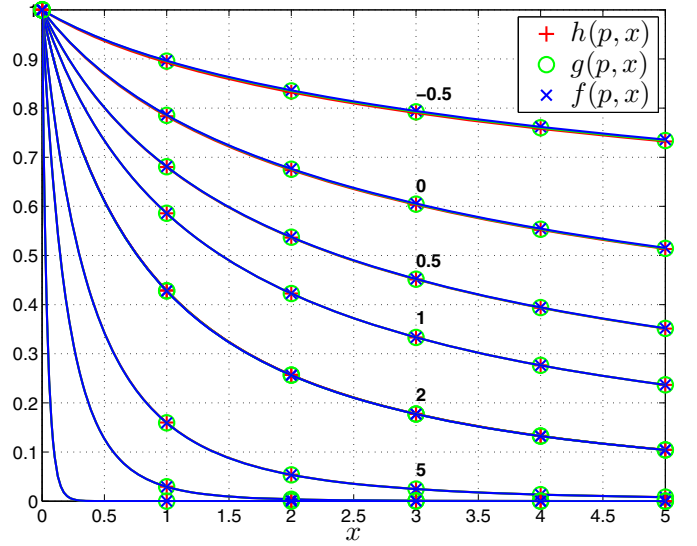


Fig. 4. For the case of $p \in \{-1/2, 0, 1/2, 1, 2, 5, 10, 50\}$, the behavior of $f(p, x)$, $g(p, x)$ and $h(p, x)$ is visualized for the range of $0 \leq x \leq 5$. Note, that for all $-1 < p \leq 1$ the double inequality $h(p, x) \leq g(p, x) \leq f(p, x)$ holds, while for all $1 \leq p < \infty$ the reverse double inequality holds. All curves are non-negative, continuous, strictly decreasing and logarithmically convex in x . They are equal at $x = 0$ and they approach zero as x approaches infinity. Furthermore, the curves are decreasing in p , as well.

than $\tau(p)$, will provide a weaker lower or a weaker upper bound of $g(p, x)$ in the corresponding domain $-1 < p \leq 1$ or $1 \leq p < \infty$, respectively. This is particularly attractive if the surrogate functions of $\tau(p)$ are of more tractable form. The bounds on $\tau(p)$ from Theorem II.2 may serve as an example for the above approach. Other appropriate bounds on $\tau(p)$ may be found in [26], [27] and [28].

In Figure 3, the numerical evaluation of the BeNaI lying between the bounds $f(p, x)$ and $h(p, x)$ is depicted for $p = 2$. As we can see, the curves are very similar and closely adjacent to one another. In Figure 4, the curves for different values of p are depicted.

In order to illustrate the quality of both bounds, we define their maximum relative errors by

$$r_f(p) = \maximize_{x \in \mathbb{R}_+} \left| \frac{f(p, x) - g(p, x)}{g(p, x)} \right| \quad (48a)$$

$$r_h(p) = \maximize_{x \in \mathbb{R}_+} \left| \frac{h(p, x) - g(p, x)}{g(p, x)} \right| \quad (48b)$$

and show the corresponding numerical results in Figure 5. For analytical results and a treatment of different types of errors in various ways, we refer our readers to [23, pp. 117-124].

For the sake of brevity, we will hereinafter write a_f , b_f , a_h and b_h instead of $a_f(p)$, $b_f(p)$, $a_h(p)$ and $b_h(p)$, unless their dependency to p needs to be emphasized. We will mathematically discuss some general properties of the curves in the next section having in mind that this is important for future applications.

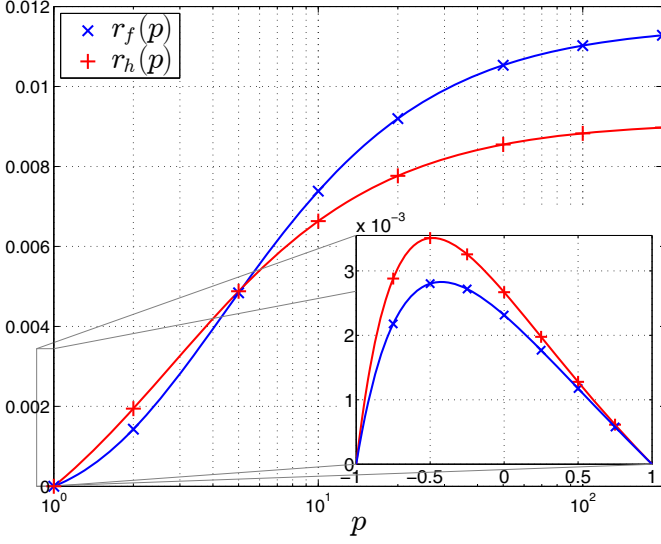


Fig. 5. The maximum relative errors $r_f(p)$ and $r_h(p)$ are visualized for $-1 \leq p \leq 1$ and $1 \leq p \leq 200$. Both $r_f(p)$ and $r_h(p)$ are limited by the upper values of 0.0035 and 0.012 respectively for $-1 \leq p \leq 1$ and $1 \leq p < \infty$.

V. FUNCTIONAL PROPERTIES OF THE BOUNDS

In this section, we investigate some properties of the bounds, that are given in Theorem IV.1 and IV.5, in order to compare them with the properties of the BeNaI discussed in [1, Section IV]. In particular, the positivity, continuity, monotonicity, convexity and series expansions of both bounds are investigated. Furthermore, we show in which cases both bounds are equal to the BeNaI. Two of the properties are given first. As an auxiliary function we use

$$\tilde{g}(p, x) := \frac{1}{(1+x)^{\frac{p}{2}}} \frac{a+1}{a+\sqrt{1+xb}} \quad (49)$$

with non-negative constants a and b , see Theorems IV.1 and IV.5 and also Lemmata IV.3 and IV.7.

Lemma V.1 For all $-1 < p < \infty$ and for all $x \geq 0$, both functions $f(p, x)$ and $h(p, x)$ are

- non-negative,
- continuous in both arguments, and
- strictly decreasing with respect to x .

Proof: The first two properties are obvious, because $a_f(p)$, $b_f(p)$, $a_h(p)$ and $b_h(p)$ are non-negative, due to Lemma IV.3 and IV.7, and because of continuity of the constituent functions. To prove monotonicity, the function $\tilde{g}(p, x)$ is differentiated, with respect to x , to obtain

$$\frac{\partial \tilde{g}(p, x)}{\partial x} = -\frac{1}{2} \tilde{g}(p, x) \left(\frac{b}{a+\sqrt{1+xb}} \frac{1}{\sqrt{1+xb}} + \frac{p}{1+x} \right). \quad (50)$$

For all $0 \leq p < \infty$, the derivative is always negative because $\tilde{g}(p, x)$ is positive as shown above, and the coefficients a and b are non-negative due to Lemma IV.3 and IV.7. Hence, $f(p, x)$ and $h(p, x)$ are strictly decreasing. Monotonicity seems to

be much harder to prove for the case $-1 < p < 0$. However, we conjecture that $f(p, x)$ and $h(p, x)$ are strictly decreasing functions of x also for $-1 < p < 0$. ■

Lemma V.2 For all $-1 < p < \infty$, both functions $f(p, x)$ and $h(p, x)$ are logarithmically convex for all arguments $x \geq 0$.

Proof: In order to prove the above statement, we have to show that the inequality

$$\tilde{g}(p, \lambda x_1 + (1-\lambda)x_2) \leq \tilde{g}^\lambda(p, x_1) \tilde{g}^{1-\lambda}(p, x_2) \quad (51)$$

holds for all $0 \leq p < \infty$, $x_1 \geq 0$, $x_2 \geq 0$ and $1 \geq \lambda \geq 0$. We use the weighted means inequality [30, p. 13, eq. 2.2.2 and p. 26, eq. 2.9.1] thrice which leads to the inequality chain

$$\begin{aligned} & [(1+x_1)^\lambda (1+x_2)^{1-\lambda}]^{\frac{p}{2}} \\ & \cdot \left[(a+\sqrt{1+bx_1})^\lambda (a+\sqrt{1+bx_2})^{1-\lambda} \right] \\ & \leq [\lambda(1+x_1) + (1-\lambda)(1+x_2)]^{\frac{p}{2}} \\ & \cdot \left[\lambda(a+\sqrt{1+bx_1}) + (1-\lambda)(a+\sqrt{1+bx_2}) \right] \\ & = [1+\lambda x_1 + (1-\lambda)x_2]^{\frac{p}{2}} \left[a+\lambda\sqrt{1+bx_1} + (1-\lambda)\sqrt{1+bx_2} \right] \\ & \leq [1+\lambda x_1 + (1-\lambda)x_2]^{\frac{p}{2}} \left[a+\sqrt{1+b\lambda x_1 + b(1-\lambda)x_2} \right]. \end{aligned} \quad (52)$$

By simple rearrangement of the last inequality and multiplying both sides by $a+1$, we conclude

$$\begin{aligned} & \frac{1}{[1+\lambda x_1 + (1-\lambda)x_2]^{\frac{p}{2}}} \frac{a+1}{a+\sqrt{1+b[\lambda x_1 + (1-\lambda)x_2]}} \\ & \leq \frac{1}{[(1+x_1)^\lambda (1+x_2)^{1-\lambda}]^{\frac{p}{2}}} \\ & \cdot \left(\frac{a+1}{a+\sqrt{1+bx_1}} \right)^\lambda \left(\frac{a+1}{a+\sqrt{1+bx_2}} \right)^{1-\lambda}, \end{aligned} \quad (53)$$

which is equivalent to inequality (51). Hence, $f(p, x)$ and $h(p, x)$ are logarithmically convex as well. Convexity seems to be much harder to prove for the case $-1 < p < 0$. However, we also conjecture that $f(p, x)$ and $h(p, x)$ are logarithmically convex in x also for $-1 < p < 0$. ■

Lemma V.3 For all $a \geq 0$, $b \geq 0$ and $-1 < p < \infty$, the expansion

$$\begin{aligned} \tilde{g}(p, x) &= 1 - \frac{b+(a+1)p}{2(a+1)} x \\ &+ \frac{(a+3)b^2 + 2(a+1)pb + (a+1)^2(p+2)p}{8(a+1)^2} x^2 - \mathcal{O}(x^3) \end{aligned} \quad (54)$$

holds at $x = 0$.

Proof: The statement is obtained by using the general definition of the Taylor expansion and straightforward calculation of the first three derivatives of the surrogate function. ■

Corollary V.4 *The Taylor expansion of $f(p, x)$ at $x = 0$ is given by*

$$f(p, x) = 1 - \frac{(p+1)(p+2)}{2(p+3)}x + \frac{4(a_f+3) + 4p(p+3) + p(p+2)(p+3)^2}{8(p+3)^2}x^2 - \mathcal{O}(x^3). \quad (55)$$

Proof: The assertion follows by incorporating (28) into (54). ■

Corollary V.5 *The Taylor expansion of $h(p, x)$ at $x = 0$ is given by*

$$h(p, x) = 1 - \frac{a_h + p(p+1)}{2(p+1)}x + \frac{(a_h+3)a_h^2 + 2p(p+1)a_h + p(p+2)(p+1)^2}{8(p+1)^2}x^2 - \mathcal{O}(x^3). \quad (56)$$

Proof: The assertion follows by incorporating (39) into (54). ■

Remark V.6 *The first two elements in the Taylor expansions of $f(p, x)$ and $g(p, x)$ are identical, while in the Taylor expansions of $h(p, x)$ and $g(p, x)$ only the both first elements are identical. Thus, $f(p, x)$ achieves a better approximation of $g(p, x)$ than $h(p, x)$ for sufficiently small x .*

Lemma V.7 *For all $a \geq 0$, $b \geq 0$ and $-1 < p < \infty$, the expansion*

$$\tilde{g}(p, x) = \frac{(a+1)}{b^{\frac{1}{2}}x^{\frac{p+1}{2}}} - \frac{(a+1)a}{bx^{\frac{p+2}{2}}} + \frac{(a+1)(2a^2 - 1 - pb)}{2b^{\frac{3}{2}}x^{\frac{p+3}{2}}} - \mathcal{O}(x^{-\frac{p+4}{2}}) \quad (57)$$

holds as $x \mapsto \infty$.

Proof: We use the general definition of asymptotic series expansion for any bounded function $\omega(x)$ which approaches a constant finite value as x approaches infinity. This definition is implicitly given by

$$\lim_{x \mapsto \infty} x^{\frac{n}{2}} \left[\omega(x) - \sum_{k=0}^n \alpha_k x^{-\frac{k}{2}} \right] = 0 \quad (58)$$

for all $n \in \mathbb{N}_0$, see [29, p. 11, Definition 1.3.3]. Now, consider the function

$$\omega(x) := x^{\frac{p+1}{2}} \tilde{g}(p, x). \quad (59)$$

Then we first consider the case of $n = 0$ and obtain the first coefficient $\alpha_0 = \frac{(a+1)}{\sqrt{b}}$ by determining the limit. Second, we increment n by one, determine the limit in (58) by applying l'Hospital's rule, and obtain the second coefficient $\alpha_1 = -\frac{(a+1)a}{b}$. The same principle is applied to obtain the coefficient α_2 and the order of the series expansion. Note that

the last three steps are straightforward, however, require intensive algebra. Finally, the sequence $\sum_{k=0}^3 \alpha_k x^{-\frac{k}{2}}$ is divided by $x^{\frac{p+1}{2}}$ which completes the proof. ■

Substituting a and b in (57) by the corresponding coefficients $a_f(p)$, $b_f(p)$, $a_h(p)$ and $b_h(p)$ yields the following two propositions.

Corollary V.8 *The asymptotic series expansion of $f(p, x)$ for $x \mapsto \infty$ is given by*

$$f(p, x) = \frac{\tau(p)}{x^{\frac{p+1}{2}}} - \frac{2\tau^2(p) - (p+3)}{2x^{\frac{p+2}{2}}} + \frac{8\tau^4(p) - 12(p+2)\tau^2(p) + (p+3)^2}{8\tau(p)x^{\frac{p+3}{2}}} - \mathcal{O}(x^{-\frac{p+4}{2}}). \quad (60)$$

Corollary V.9 *The asymptotic series expansion of $h(p, x)$ for $x \mapsto \infty$ is given by*

$$h(p, x) = \frac{\tau(p)}{x^{\frac{p+1}{2}}} - \frac{(p+1)}{x^{\frac{p+2}{2}}} + \frac{-\tau^4(p) + (p+2)\tau^2(p) + (p+1)^2}{2\tau(p)x^{\frac{p+3}{2}}} - \mathcal{O}(x^{-\frac{p+4}{2}}). \quad (61)$$

Remark V.10 *Only the both first elements in the asymptotic series expansions of $f(p, x)$ and $g(p, x)$ are identical, while in the asymptotic series expansions of $h(p, x)$ and $g(p, x)$ the first two elements are identical. Thus, $h(p, x)$ achieves a better approximation of $g(p, x)$ than $f(p, x)$ for sufficiently large x .*

As mentioned at the beginning of Section IV, the coefficients e_1 , e_2 and e_3 in (24) are chosen such that Remark V.6 and V.10 are fulfilled. It is near at hand to aim at choosing the coefficients in an optimal way, namely to minimize the difference between the bounds and the BeNaI. However, because of the analytical complexity this seems to be out of reach.

The above statements show the general and asymptotic behavior of the bounds, while the following ones describe the relationship to the BeNaI. In particular, it is shown in which cases the bounds and the BeNaI are equal.

Corollary V.11 *If $x = 0$, it holds for all $-1 < p < \infty$ that*

$$f(p, 0) = h(p, 0) = g(p, 0) = 1. \quad (62)$$

Proof: The equality $g(p, 0) = 1$ is given by (19a). By inserting $x = 0$ into (25) and (36), we obtain the equalities $f(p, 0) = 1$ and $h(p, 0) = 1$, respectively. ■

Corollary V.12 *For all $-1 < p < \infty$, the functions $f(p, x)$, $h(p, x)$ and $g(p, x)$ approach the asymptote $\tau(p)x^{-\frac{p+1}{2}}$ as x approaches infinity. Thus, it follows that*

$$\lim_{x \mapsto \infty} f(p, x) = \lim_{x \mapsto \infty} h(p, x) = \lim_{x \mapsto \infty} g(p, x) = 0. \quad (63)$$

Proof: The asymptote $\tau(p)x^{-\frac{p+1}{2}}$ follows from the asymptotic expansions, which are stated in equation (19b),

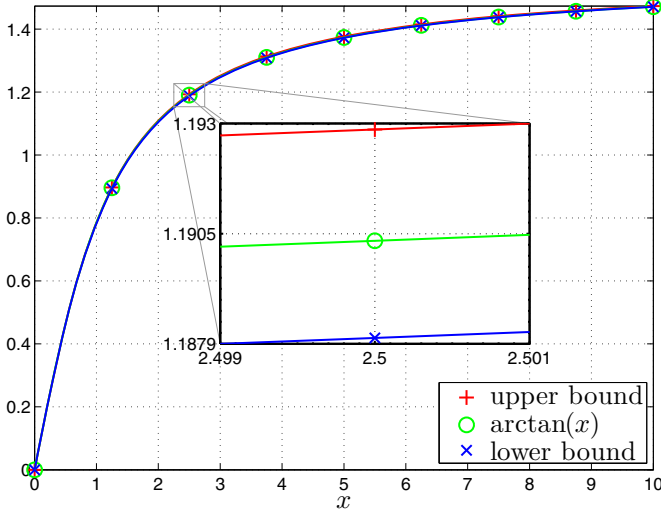


Fig. 6. The inverse tangent function and its bounds from Corollary V.14 are visualized for the range of $0 \leq x \leq 10$. The curves are closely adjacent to one another such that without magnification the differences are not really visible. The curves are equal at zero and approach the same limit as $|x|$ approaches infinity. Their maximum relative errors are approximately less than 0.23% and 0.27% which shows how sharp both bounds are.

Corollary V.8 and Corollary V.9. From this asymptote we derive the limit which tends toward zero. ■

Corollary V.13 *In case of $p \mapsto -1$ and for all $x \geq 0$, we observe the equality*

$$f(-1, x) = h(-1, x) = \lim_{p \rightarrow -1} g(p, x) = 1. \quad (64)$$

Proof: By using the identity $\tau(-1) = 1$ from (13), we obtain the equalities $a_f(-1) = 0$ and $a_h(-1) = 0$ from (26a) and (37a), respectively. By inserting $a_f(-1)$ and $a_h(-1)$ into (34) and (45), respectively, we derive equation (64) by considering (18c). ■

Corollary V.14 *If $p = 0$ and $x \geq 0$, the double inequality*

$$\frac{\pi^2 x}{4 + \sqrt{(\pi^2 - 4)^2 + 4\pi^2 x^2}} \leq \arctan(x) \leq \frac{\pi^2 x}{\pi^2 - 6 + 2\sqrt{9 + \pi^2 x^2}} \quad (65)$$

holds. For all $x < 0$, the reverse double inequality holds.

Proof: The double inequality (65) follows from equation (18b), inequalities (27) and (38), and by replacing x with x^2 . ■

The double inequality in Corollary V.14 is tight as can be seen from Figure 6. The maximum relative errors of the bounds, related to the inverse tangent function, are approximately less than 0.23% and 0.27%. Inequality (65) is obtained as a side result of the general approach in this work. A more detailed discussion is included in [31].

As shown in the representation [1, eq. 29b], the fractions $\frac{a_f(p)+1}{a_f(p)+\sqrt{1+x}b_f(p)}$ and $\frac{a_h(p)+1}{a_h(p)+\sqrt{1+x}b_h(p)}$ may also be

seen as approximations of the hypergeometric function ${}_2F_1(\frac{1}{2}, 1; \frac{p+3}{2}; -x)$. Thus, they can be used as bounds for other functions that can be described in terms of the hypergeometric function and its transformations.

Corollary V.15 *If $p = 1$, it holds for all $x \geq 0$ that*

$$f(1, x) = h(1, x) = g(1, x) = \frac{2}{1 + x + \sqrt{1 + x}}. \quad (66)$$

Proof: By using the identity $\tau(1) = 2$ from (13), we obtain the equalities $a_f(1) = 1$ and $a_h(1) = 1$ from (26a) and (37a), respectively. By inserting $a_f(1)$ and $a_h(1)$ into (34) and (45), respectively, we derive equation (66) by considering (18a). ■

In the current section, we have shown some mathematical properties of the bounds which are identical to those of the BeNaI. Unfortunately, we could not prove the monotonicity and the convexity of the bounds for the case $-1 < p < 0$ and thus we leave the proof as an open problem.

VI. DISCUSSION OF RESULTS AND CONCLUSION

As shown in (17) from our previous work [1, Theorem 6], we have found new representations for the Beta-Nakagami integral (BeNaI) with considerable consequences. On the one hand, equation (17) does not include any special functions and is thus simpler to handle than (16). On the other hand, by the new representation (17) we have been able to deduce a proper class of functions for determining accurate bounds in closed-form for the BeNaI.

As stated in Theorem IV.1 and IV.5, we have presented two novel bounds on the BeNaI for the whole range of parameters $-1 < p < \infty$ and $x \geq 0$. Both lower and upper bound are dependent on their coefficients and it is possible to modify the coefficients in order to adapt the bounds for certain applications. This shows that the proposed bounds are scalable and are thus universally useful. In addition, we have illustrated that both bounds are sharp and accurate over a wide range of parameters. Their maximum relative errors over the whole range of parameters are approximately less than 1.2%.

In addition, we have shown in [1, Section IV] and in Section V that the BeNaI and both bounds have some famous mathematical properties in common. In particular, they are non-negative, continuous, strictly decreasing and logarithmically convex with respect to x . All series expansions of the BeNaI and both bounds are also derived, especially for the limits $x \mapsto 0$ and $x \mapsto \infty$ in order to understand their asymptotic behavior. By studying these properties, we have established the existence of certain cases where both bounds and the BeNaI are equal. This reinforces the decision about the chosen class of functions from (24) for generating the bounds in Theorem IV.1 and IV.5.

We believe that the same proposed methods, for investigation of the average error probability (AEP) in communication over Nakagami-distributed fading channels, are also applicable for the investigation of the AEP over Rice-distributed fading channels and devote this investigation for future works.

VII. OPEN PROBLEMS

We have not found any short proof in order to show the monotonicity and the convexity of the bounds $f(p, x)$ and $h(p, x)$ for the range $-1 < p < 0$. Properties of *completely monotonic* functions, see [32], could be of help to accomplish this task, which we leave for future research. For the time being, both properties are claimed as conjectures.

Conjecture VII.1 *For all $-1 < p < 0$ and for all $x \geq 0$, both $f(p, x)$ and $h(p, x)$ are strictly decreasing functions of x .*

Conjecture VII.2 *For all $-1 < p < 0$ and for all $x \geq 0$, both $f(p, x)$ and $h(p, x)$ are convex functions of x .*

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