Power, Rate and QoS Control for Impulse Radio

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***Abstract*— Because of its limitations due to interference, ultra-wideband impulse radio needs a careful investigation of the signal-to-interference ratio (SIR). This paper contributes to understanding the theoretical foundations of the SIR effects in multi-user impulse radio systems. Two basic aspects are investigated. First, the set of admissible power allocations to maintain certain quality-of-service thresholds is described. Secondly, the geometry of the set of feasible reciprocal bit rates is characterized. Both sets turn out to have certain monotonicity and convexity properties. We furthermore discuss the case that the pulse repetition time is used as a global parameter to achieve an admissible power allocation at the price of proportionally reduced quality-of-service parameters. Techniques from analyzing code division multiple access systems are generalized and applied to impulse radio in this paper.**

I. Introduction

Ultra-wideband impulse radio (UWB-IR) is a promising technology for communication at extremely high data rates with low power consumption, mainly over short distances. Impulse radio operates in the spectrum from near DC to a few gigahertz, a highly populated frequency band, and hence must content with a variety of interfering signals. Furthermore, low intra-system interference from other users must be ensured. Applying impulse radio as a multiuser system necessitates careful power control for a fair radio resource sharing, see [1], [2].

Since the system is interference limited, the signal-to-interference ratio (SIR) plays a prominent role for assessing the quality of transmission and the capacity of the system, see [3]. Once the propagation environment is known, the SIR boils down to a rational function of essentially two variables, the power allocation and the binary bit rate.

In this paper, we choose two complementary approaches to describe IR system performance. First, we investigate the unrestricted set of power adjustments for a community of users such that nobody's SIR falls short of an individually chosen threshold. It turns out that this set has nice geometrical properties. It has a uniformly minimal point which is componentwise increasing as the QoS requirements increase.

Secondly, we consider the set of all reciprocal bit rates that can be supported by some admissible power assignment. This set turns out to be convex and monotonic, provided power assignments form a convex set and are such that any reduced power allocation is admissible in case its predecessor is.

Our approach stresses the analogy of important modeling aspects for both impulse radio and code division multiple access (CDMA). Power control and feasible quality of service parameters can be treated for both systems in a similar setup. We generalize and apply analytical techniques from CDMA which can be found in [4], [5], [6]. We start with a short system model overview in Section II. In Section III we deal with the set of admissible power adjustments. Section IV characterizes all feasible reciprocal bit rates, which can be supported by some power allocation from a limited set. Finally, we use the pulse repetition time as a global parameter for access control to achieve a power allocation in a constrained set at the price of proportionally reduced QoS parameters in Section V. We conclude with a short summary in Section VI.

II. System Model

We start by briefly recalling some basic facts about ultra wideband (UWB) radio, or as alternatively referred to, impulse radio (IR). A short general introduction is given in [1]. In impulse radio systems, extremely short pulses are transmitted (0.1 to 1.5 ns). A typical pulse, named monocycle, is the Gaussian pulse $g(t)$, see [3].

We consider a multi-user UWB system with K users. A time-hopping code is used in order to accommodate multiple users in UWB systems. For transmitting binary symbols pulse position modulation (PPM) is used. A sequence of $N_{S,i}$ not shifted pulses is transmitted for the symbol 0, while the symbol 1 is transmitted by a sequence of $N_{S,i}$ pulses shifted by amount δ for user i. The resulting transmitted signal of user i is

$$s_i^k(t) = A_k \sum_{j=-\infty}^{\infty} g(t - jT_f - c_j^k T_c - \delta d^k_{\lfloor j/N_{S,i}\rfloor})$$

where

- A_k is the pulse amplitude,
- T_f is the pulse repetition time interval,
- $c_j^{(k)}$ is the time hopping code for the k-th user,
- T_c is the time shift defined for the hopping code,
- d_i^k is the i-th symbol of the k-th user.

By assuming an AWGN channel model with perfect timing, the SIR of user i is

$$\mathrm{SIR}_i = \frac{A_i^2 m_p^2 N_{s,i}^2 G_{ii}}{\tau_a^2 \sum_{j\neq i} A_j^2 G_{ij} + N_{0,i} m_p}$$

where

- G_{ij} is the path gain from user j to the receiver of user i,

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- $m_p = \int_{-\infty}^{\infty} g(t)v(t)dt$, with $v(t) = g(t) - g(t-\delta)$
- $\tau_a^2 = \frac{1}{T_f}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} g(t-s)v(t)dt\right)^2 ds$ is the interference power resulting from one user,
- $N_{S,i}$ is the number of reduplicate pulses for one bit for user i,
- $N_{0,i}$ is the thermal noise.

The SIR_i defined above can also be expressed as a function of the transmission powers $\boldsymbol{p} = (p_1, \ldots, p_K)$ by introducing the following relationships

$$\begin{aligned} p_i &= \frac{1}{T_f}\int_0^{T_f} A_i^2 g^2(t)dt = \frac{A_i^2 E_W}{T_f} \\ E_W &= \int_0^{T_f} g^2(t)dt,\ \tau_b^2 = \frac{\tau_a^2}{m_p^2}, \\ \eta_i &= \frac{N_{0,i}E_W}{m_p} \text{ the background noise energy} \\ r_i &= \frac{1}{N_{S,i}T_f} \text{ the binary bit rate of user } i. \end{aligned}$$

The SIR_i for user i can then be written as, cf. [2],

$$\mathrm{SIR}_i(\boldsymbol{p}, \boldsymbol{r}) = \frac{G_{ii}p_i}{r_i(T_f\tau_b^2\sum_{j\neq i} G_{ij}p_j + \eta_i)},$$

where $\boldsymbol{p} = (p_1, \ldots, p_K)'$ and $\boldsymbol{r} = (r_1, \ldots, r_K)'$ denote the vector of power adjustments and binary rates. Integrating the system constants T_f and τ_b^2 into the path gains by setting

$$A_{ii} = G_{ii},\ A_{ij} = T_f\tau_b^2 G_{ij}, i \neq j,$$

yields

$$\mathrm{SIR}_i(\boldsymbol{p}, \boldsymbol{r}) = \frac{A_{ii}p_i}{r_i(\sum_{j\neq i} A_{ij}p_j + \eta_i)}. \tag{1}$$

Formula (1) is similar to the signal-to-interference ratio of synchronous multiuser CDMA systems with K users and processing gain N, see [7]. Let $\boldsymbol{s}_i \in \mathbb{R}^N$, $i = 1, \ldots, K$, the N-dimensional signature sequence of user i. Denote by G_{ij} the fixed path gain from user j to the assigned base station of user i. Suppose that the symbol of user i is decoded using a linear receiver represented by some vector $\boldsymbol{c}_i \in \mathbb{R}^N$. The signal-to-interference ratio of user i is then given as

$$\mathrm{SIR}_i(\boldsymbol{p}) = \frac{G_{ii}(\boldsymbol{c}_i'\boldsymbol{s}_i)^2 p_i}{\sum_{j\neq i} G_{ij}(\boldsymbol{c}_i'\boldsymbol{s}_j)^2 p_j + \sigma^2(\boldsymbol{c}_i'\boldsymbol{c}_i)},$$

where σ^2 denotes the variance of the additive Gaussian noise and $\boldsymbol{p} = (p_1, \ldots, p_K)$ the vector of transmit powers.

Summarizing the known channel and receiver effects into

$$A_{ij} = G_{ij}(\boldsymbol{c}_i'\boldsymbol{s}_j)^2 \text{ and } \eta_i' = \sigma^2(\boldsymbol{c}_i'\boldsymbol{c}_i)$$

gives equation (1) with $r_i = 1$ for all $i = 1, \ldots, K$.

III. Power and Rate Allocation

In this section we analyze the ramification of both the power p_i and the data rate r_i. Our starting point is equation (1).

Given QoS requirements $\gamma_1, \ldots, \gamma_K$ for each user the power region $\mathcal{P}_{\mathrm{SIR}}^{\mathrm{IR}}$ of an ultra wideband radio system with K users is defined as

$$\mathcal{P}_{\mathrm{SIR}}^{\mathrm{IR}}(\boldsymbol{\gamma}, \boldsymbol{r}) = \left\{\boldsymbol{p} \geq \boldsymbol{0} \mid \mathrm{SIR}_i(\boldsymbol{p}, \boldsymbol{r}) \geq \gamma_i,\ i = 1, \ldots, K\right\}. \tag{2}$$

The inequalities defining (2) can be rewritten as a system of linear inequalities. For this purpose write $\boldsymbol{B} = (b_{ij})_{i,j=1}^K$, with

$$b_{ij} = \begin{cases} A_{ij}/A_{ii}, & i \neq j, \\ 0, & i = j, \end{cases} \tag{3}$$

and

$$\boldsymbol{\tau} = (\tau_1, \ldots, \tau_K)^{\mathsf{T}},\ \text{where } \tau_i = \eta_i/A_{ii}. \tag{4}$$

Then for every $\boldsymbol{p} > \boldsymbol{0}$ it holds that $\boldsymbol{p} \in \mathcal{P}_{\mathrm{SIR}}^{\mathrm{IR}}(\boldsymbol{\gamma}, \boldsymbol{r})$ if and only if

$$[\boldsymbol{I} - \boldsymbol{\Gamma RB}]\,\boldsymbol{p} \geq \boldsymbol{\Gamma R\tau}, \tag{5}$$

where $\boldsymbol{\Gamma} = \mathrm{diag}(\boldsymbol{\gamma})$ and $\boldsymbol{R} = \mathrm{diag}(\boldsymbol{r})$ denote the matrices with diagonal entries γ_i and r_i, respectively, and non diagonal entries equal to zero.

For convenience of notation we quote the following result from [6]. It deals with solutions of the equation

$$[\boldsymbol{I} - \boldsymbol{A}]\boldsymbol{x} = \boldsymbol{c} \tag{6}$$

when $\boldsymbol{A}$ is a non-negative but not necessarily irreducible matrix. The proof given in [6] is direct and self-contained, and it extends the Perron-Frobenius theory by avoiding the assumption of irreducibility.

Let $\rho(\boldsymbol{A})$ denote the spectral radius of the square matrix $\boldsymbol{A}$, i.e., the smallest radius of a disc centered at the origin in the complex plane that covers all eigenvalues of $\boldsymbol{A}$, or

$$\rho(\boldsymbol{A}) = \max\{|\lambda| \mid \lambda \text{ is an eigenvalue of } \boldsymbol{A}\}.$$

Proposition 1 *Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be non-negative.*

a) *If there are $\boldsymbol{x} > \boldsymbol{0}$, $\boldsymbol{c} > \boldsymbol{0}$ satisfying* (6)*, then $\rho(\boldsymbol{A}) < 1$.*
b) *If $\rho(\boldsymbol{A}) < 1$, then $\boldsymbol{I} - \boldsymbol{A}$ is non-singular and for every $\boldsymbol{c} > \boldsymbol{0}$, the unique solution $\boldsymbol{x} \in \mathbb{R}^n$ of* (6) *is positive.*
c) *If $\rho(\boldsymbol{A}) < 1$, then for every $\boldsymbol{c} \geq \boldsymbol{0}$, the unique solution $\boldsymbol{x} \in \mathbb{R}^n$ of* (6) *is non-negative.*
d) *If $\boldsymbol{c} > \boldsymbol{0}$ and there exists $\boldsymbol{y} > \boldsymbol{0}$ such that $[\boldsymbol{I} - \boldsymbol{A}]\boldsymbol{y} \geq \boldsymbol{c}$, then* (6) *has a unique solution $\boldsymbol{x}$ and $\boldsymbol{0} < \boldsymbol{x} \leq \boldsymbol{y}$.*

If system (5) has a solution $\boldsymbol{p} > 0$, then there is a unique solution $\boldsymbol{p}^* \leq \boldsymbol{p}$ satisfying

$$[\boldsymbol{I} - \boldsymbol{\Gamma RB}]\,\boldsymbol{p}^* = \boldsymbol{\Gamma R\tau}, \tag{7}$$

as follows from Proposition 1. Moreover, for any given $\boldsymbol{\gamma} > \boldsymbol{0}$ and $\boldsymbol{r} > \boldsymbol{0}$, the equation $[\boldsymbol{I} - \boldsymbol{\Gamma RB}]\,\boldsymbol{p} = \boldsymbol{\Gamma R\tau}$ has a positive solution $\boldsymbol{p}$ if and only if the spectral radius $\rho(\boldsymbol{\Gamma RB}) < 1$, and in that case, the solution is unique. Denote it by $\boldsymbol{p}^*(\boldsymbol{\gamma}, \boldsymbol{r})(p_1^*(\boldsymbol{\gamma}, \boldsymbol{r}), \ldots, p_K^*(\boldsymbol{\gamma}, \boldsymbol{r}))'$. Thus

$$\boldsymbol{p}^*(\boldsymbol{\gamma}, \boldsymbol{r}) = [\boldsymbol{I} - \boldsymbol{\Gamma RB}]^{-1}\,\boldsymbol{\Gamma R\tau} \tag{8}$$

with all components positive.

Summarizing our results so far, we see that there is a unique componentwise minimum power allocation in $\mathcal{P}_{\mathrm{SIR}}^{\mathrm{IR}}$, provided that the power region is nonempty.

Proposition 2 *If $\mathcal{P}^{\mathrm{IR}}_{\mathrm{SIR}}(\boldsymbol{\gamma},\boldsymbol{r}) \neq \emptyset$, then there is a unique power allocation $\boldsymbol{p}^* = \boldsymbol{p}(\boldsymbol{\gamma},\boldsymbol{r})$ such that*

$$\mathrm{SIR}_i(\boldsymbol{p}^*,\boldsymbol{r}) = \gamma_i \quad \text{for all } i = 1,\dots,K \text{ and}$$
$$\boldsymbol{p}^* \leq \boldsymbol{p} \quad \text{for all } \boldsymbol{p} \in \mathcal{P}^{\mathrm{IR}}_{\mathrm{SIR}}(\boldsymbol{\gamma},\boldsymbol{r}).$$

The uniformly minimal point $\boldsymbol{p}^*(\boldsymbol{\gamma},\boldsymbol{r}) \in \mathcal{P}^{\mathrm{IR}}_{\mathrm{SIR}}(\boldsymbol{\gamma},\boldsymbol{r})$ is of particular interest since it requires minimal energy while maintaining the SIR demands $\boldsymbol{\gamma} = (\gamma_1,\dots,\gamma_n)'$ and binary rates $\boldsymbol{r} = (r_1,\dots,r_K)'$ of all users. In the following we deal with the behavior of $\boldsymbol{p}^*(\boldsymbol{\gamma},\boldsymbol{r})$ as a function of $\boldsymbol{\gamma}$ and $\boldsymbol{r}$.

Proposition 3 *The function $\boldsymbol{p}^*(\boldsymbol{\gamma},\boldsymbol{r})$ is monotonically increasing, i.e., if $\mathcal{P}^{\mathrm{IR}}_{\mathrm{SIR}}(\boldsymbol{\gamma}^{(2)},\boldsymbol{r}^{(2)}) \neq \emptyset$ and $\boldsymbol{\gamma}^{(1)} \leq \boldsymbol{\gamma}^{(2)}$ and $\boldsymbol{r}^{(1)} \leq \boldsymbol{r}^{(2)}$, then $\boldsymbol{p}^*(\boldsymbol{\gamma}^{(1)},\boldsymbol{r}^{(1)}) \leq \boldsymbol{p}^*(\boldsymbol{\gamma}^{(2)},\boldsymbol{r}^{(2)})$. Furthermore, $\boldsymbol{p}^*(\boldsymbol{\gamma},\boldsymbol{r}) \to \boldsymbol{0}$ as $\boldsymbol{\gamma} \to \boldsymbol{0}$ or $\boldsymbol{r} \to \boldsymbol{0}$.*

Proof: From Proposition 1 a) it follows that $\rho(\boldsymbol{\Gamma}^{(2)}\boldsymbol{R}^{(2)}\boldsymbol{B}) < 1$. Hence, expanding representation (8) in a von Neumann series gives

$$\begin{aligned}
\boldsymbol{p}^*(\boldsymbol{\gamma}^{(1)}\boldsymbol{r}^{(1)}) &= [\boldsymbol{I} - \boldsymbol{\Gamma}^{(1)}\boldsymbol{R}^{(1)}\boldsymbol{B}]^{-1}\boldsymbol{\Gamma}^{(1)}\boldsymbol{R}^{(1)}\boldsymbol{\tau} \\
&= \sum_{l=0}^{\infty}(\boldsymbol{\Gamma}^{(1)}\boldsymbol{R}^{(1)}\boldsymbol{B})^l\boldsymbol{\Gamma}^{(1)}\boldsymbol{R}^{(1)}\boldsymbol{\tau} \\
&\leq \sum_{l=0}^{\infty}(\boldsymbol{\Gamma}^{(2)}\boldsymbol{R}^{(2)}\boldsymbol{B})^l\boldsymbol{\Gamma}^{(2)}\boldsymbol{R}^{(2)}\boldsymbol{\tau} \\
&= [\boldsymbol{I} - \boldsymbol{\Gamma}^{(2)}\boldsymbol{R}^{(2)}\boldsymbol{B}]^{-1}\boldsymbol{\Gamma}^{(2)}\boldsymbol{R}^{(2)}\boldsymbol{\tau} = \boldsymbol{p}^*(\boldsymbol{\gamma}^{(2)},\boldsymbol{r}^{(2)}),
\end{aligned}$$

which proves the first assertion.

It is immediate from (8) that $\boldsymbol{p}^*(\boldsymbol{\gamma},\boldsymbol{r}) \to \boldsymbol{0}$ as $\boldsymbol{\gamma} \to \boldsymbol{0}$ or $\boldsymbol{r} \to \boldsymbol{0}$. Observe that $[\boldsymbol{I} - \boldsymbol{\Gamma}\boldsymbol{R}\boldsymbol{B}]^{-1}$ exists in a sufficiently small neighborhood of $\boldsymbol{0}$. ∎

We now analyze the geometrical properties of the power region $\mathcal{P}^{\mathrm{IR}}_{\mathrm{SIR}}(\boldsymbol{\gamma},\boldsymbol{r})$, but first quote two basic definitions.

A set $\mathcal{C}$ is called log-convex, if for any $\boldsymbol{p}^{(1)}$, $\boldsymbol{p}^{(2)} \in \mathcal{C}$ and any $0 \leq \alpha \leq 1$ the point

$$\boldsymbol{p}^{(\alpha)} = \boldsymbol{p}^{(1)^{\alpha}}\boldsymbol{p}^{(2)^{1-\alpha}} \in \mathcal{C},$$

where powers $\boldsymbol{p}^{\alpha} = (p_1^{\alpha},\dots,p_K^{\alpha})$ are applied componentwise. Taking logarithms componentwise gives

$$\log \boldsymbol{p}^{(\alpha)} = \alpha \log \boldsymbol{p}^{(1)} + (1-\alpha)\log \boldsymbol{p}^{(2)}$$

which means that the set $\mathcal{C}$ is convex in logarithmic scale.

A set $\mathcal{C}$ is said to be a cone if $\boldsymbol{p} \in \mathcal{C}$ implies that $\alpha\boldsymbol{p} \in \mathcal{C}$ for all $\alpha \geq 0$.

Proposition 4 *The shifted power region $\mathcal{P}^{\mathrm{IR}}_{\mathrm{SIR}}(\boldsymbol{\gamma},\boldsymbol{r}) - \boldsymbol{p}^*(\boldsymbol{\gamma},\boldsymbol{r})$ is a closed convex cone in $\mathbb{R}^n$. Moreover, $\mathcal{P}^{\mathrm{IR}}_{\mathrm{SIR}}(\boldsymbol{\gamma},\boldsymbol{r})$ is log-convex.*

The proof of Proposition 4 follows immediately from according assertions for the power region in CDMA, see [4], [5].

IV. Feasible Bit Rates

We now deal with the question which binary bit rates r_i, $i = 1,\dots,K$, can be supported by impulse radio in a multiuser environment where users may select a power adjustment from a possibly bounded set $\mathcal{P}$. Since $r_i = \frac{1}{N_{S,i}\,T_f}$ for pulse repetition times $N_{S,i}$ and time interval T_f, we rephrase this question in terms of $N_{S,i}$.

Let $\mathcal{P} \subset \mathbb{R}^n$ denote the set of admissible power adjustments. We assume that $\mathcal{P}$ is convex and closed under simultaneously turning power down, i.e.,

$$\text{if } \boldsymbol{p} \in \mathcal{P} \text{ and } \boldsymbol{0} < \boldsymbol{q} \leq \boldsymbol{p} \text{ then } \boldsymbol{q} \in \mathcal{P}. \tag{9}$$

Special cases of such constraints are the ℓ_t-norms as

$$\mathcal{P} = \{(p_1,\dots,p_K) \mid \sum_i p_i^t \leq p_{\max},\ 1 \leq t \leq \infty\}$$

which includes total power restrictions $\sum_i p_i \leq p_{\max}$ for $t = 1$, particularly dealt with in [8]. Individual restrictions $p_i \leq p_{\max,i}$, $i = 1,\dots,K$ also belong to this class, see [9].

Requiring quality of service constraints in terms of the signal-to-interference ratio (1) reads as

$$\mathrm{SIR}_i(\boldsymbol{p},\boldsymbol{r}) = \frac{A_{ii}p_i}{r_i(\sum_{j\neq i}A_{ij}p_j + \eta_i)} \geq \gamma_i$$

for all $i = 1,\dots,K$, or equivalently, by substituting $r_i = \frac{1}{N_{S,i}T_f}$

$$\frac{A_{ii}p_i}{\sum_{j\neq i}A_{ij}p_j + \eta_i} \geq \frac{\gamma_i}{N_{S,i}\,T_f}.$$

We relax the problem of describing the set of discrete pulse repetition numbers by introducing continuous variables $n_i \geq 0$ substituting $N_{S,i}$, in the sequel called *fractional pulse repetition numbers*. Let $\boldsymbol{n} = (n_1,\dots,n_K)' \in \mathbb{R}^K$ with $n_i = N_{S,i}$.

The set of feasible fractional repetition numbers is then defined as

$$\begin{aligned}
\mathcal{F}_{\mathcal{P}} = \Big\{&\boldsymbol{n} = (n_1,\dots,n_K)' > \boldsymbol{0} \mid \\
&\exists \boldsymbol{p} \in \mathcal{P} : \frac{A_{ii}p_i}{\sum_{j\neq i}A_{ij}p_j + \eta_i} \geq \frac{\gamma_i}{n_i\,T_f}\ \forall i\Big\}.
\end{aligned} \tag{10}$$

The function

$$\psi(\gamma_i, x) = \frac{\gamma_i}{x\,T_f},\ x > 0,$$

is obviously log-convex with respect to x, which means, by definition, that $\log\psi(\gamma_i,x)$ is convex. Writing the diagonal matrix

$$\boldsymbol{\Psi}(\boldsymbol{\gamma},\boldsymbol{n}) = \mathrm{diag}\big(\psi(\gamma_1,n_1),\dots,\psi(\gamma_K,n_K)\big)$$

and using Proposition 1 yields the following representation.

$$\mathcal{F}_{\mathcal{P}} = \big\{\boldsymbol{n} > \boldsymbol{0} \mid \exists \boldsymbol{p} \in \mathcal{P} : [\boldsymbol{I} - \boldsymbol{\Psi}(\boldsymbol{\gamma},\boldsymbol{n})\boldsymbol{B}]\boldsymbol{p} = \boldsymbol{\Psi}(\boldsymbol{\gamma},\boldsymbol{n})\boldsymbol{\tau}\big\}.$$

with $\boldsymbol{B}$ and $\boldsymbol{\tau}$ defined in (3) and (4), respectively. Analogously let

$$\mathcal{F} = \big\{\boldsymbol{n} > \boldsymbol{0} \mid \exists \boldsymbol{p} > \boldsymbol{0} : [\boldsymbol{I} - \boldsymbol{\Psi}(\boldsymbol{\gamma},\boldsymbol{n})\boldsymbol{B}]\boldsymbol{p} = \boldsymbol{\Psi}(\boldsymbol{\gamma},\boldsymbol{n})\boldsymbol{\tau}\big\}.$$

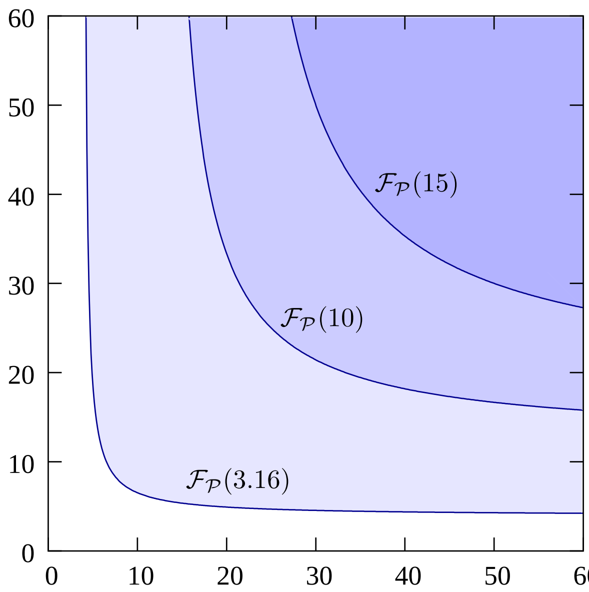

Fig. 1. The sets $\mathcal{F}_{\mathcal{P}}$ for SIR requirements $\gamma_1 = \gamma_2 = 3.16, 10.0, 15.0$ (shaded from light to dark) with total power restricted by $p_1 + p_2 \leq 10^{-4}$.

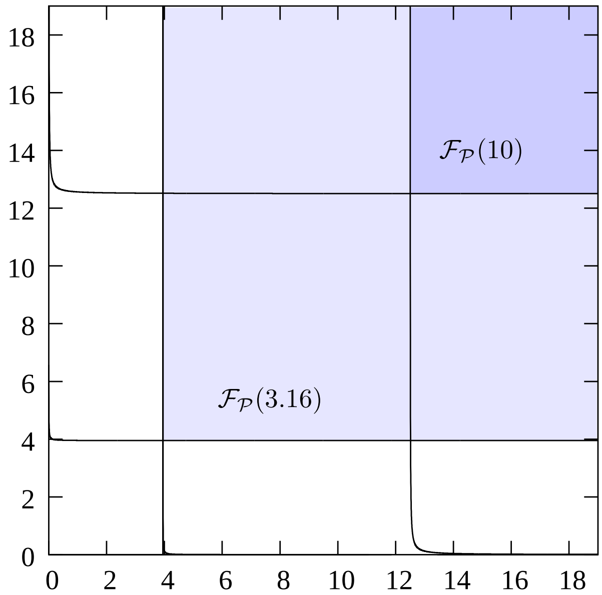

Fig. 2. The sets $\mathcal{F}_{\mathcal{P}}$ for SIR requirements 3.16, 10 with bounded maximum power, $\max\{p_1, p_2\} \leq 10^{-4}$.

denote the set of feasible fractional repetition numbers with unlimited power. It is easy to see that $\mathcal{F}$ is independent of parameter T_f which, however, fails to hold for $\mathcal{F}_{\mathcal{P}}$. If $\boldsymbol{n} \in \mathcal{F}$, then according to Proposition 1 there exists a unique $\boldsymbol{p}^*(\boldsymbol{n}) > 0$ solving the system of equations $[\boldsymbol{I} - \boldsymbol{\Psi}(\boldsymbol{\gamma}, \boldsymbol{n})\boldsymbol{B}]\boldsymbol{p} = \boldsymbol{\Psi}(\boldsymbol{\gamma}, \boldsymbol{n})\boldsymbol{\tau}$. Further, regarding monotonous SIR requirements, the set $\mathcal{F}_{\mathcal{P}}$ has the following property.

Proposition 5 *Let $\boldsymbol{\gamma}^{(1)} \leq \boldsymbol{\gamma}^{(2)}$ be SIR requirements. Then it holds that $\mathcal{F}_{\mathcal{P}}(\boldsymbol{\gamma}^{(2)}) \subseteq \mathcal{F}_{\mathcal{P}}(\boldsymbol{\gamma}^{(1)})$, with the obvious notation $\mathcal{F}_{\mathcal{P}}(\boldsymbol{\gamma}^{(j)}) = \{\boldsymbol{n} > \mathbf{0} \mid \exists \boldsymbol{p} \in \mathcal{P} : [\boldsymbol{I} - \boldsymbol{\Psi}(\boldsymbol{\gamma}^{(j)}\boldsymbol{n})\boldsymbol{B}]\boldsymbol{p} = \boldsymbol{\Psi}(\boldsymbol{\gamma}^{(j)}, \boldsymbol{n})\boldsymbol{\tau}\}$.*

Proof: From $\boldsymbol{\gamma}^{(1)} \leq \boldsymbol{\gamma}^{(2)}$ it follows that $\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(1)}, \boldsymbol{n}) \leq \boldsymbol{\Psi}(\boldsymbol{\gamma}^{(2)}, \boldsymbol{n})$, and thus $\rho(\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(1)}, \boldsymbol{n})) \leq \rho(\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(2)}, \boldsymbol{n}))$. By Proposition 1 $(\boldsymbol{I} - \boldsymbol{\Psi}(\boldsymbol{\gamma}^{(1)}, \boldsymbol{n})\boldsymbol{B})^{-1}$ exists whenever $(\boldsymbol{I} - \boldsymbol{\Psi}(\boldsymbol{\gamma}^{(2)}, \boldsymbol{n})\boldsymbol{B})^{-1}$ exists.

It remains to show that $\boldsymbol{p}^{(1)} \in \mathcal{P}$ whenever $\boldsymbol{p}^{(2)} \in \mathcal{P}$ with $\boldsymbol{p}^{(j)} = (\boldsymbol{I} - \boldsymbol{\Psi}(\boldsymbol{\gamma}^{(j)}, \boldsymbol{n})\boldsymbol{B})^{-1}\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(j)}, \boldsymbol{n})\boldsymbol{\tau}, j = 1, 2$. Expanding $\boldsymbol{p}^{(j)}$ in a von Neumann series gives

$$\begin{aligned}
\boldsymbol{p}^{(1)} &= (\boldsymbol{I} - \boldsymbol{\Psi}(\boldsymbol{\gamma}^{(1)}, \boldsymbol{n})\boldsymbol{B})^{-1}\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(1)}, \boldsymbol{n})\boldsymbol{\tau} \\
&= \sum_{l=0}^{\infty}(\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(1)}, \boldsymbol{n})\boldsymbol{B}))^l\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(1)}, \boldsymbol{n})\boldsymbol{\tau} \\
&\leq \sum_{l=0}^{\infty}(\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(2)}, \boldsymbol{n})\boldsymbol{B}))^l\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(2)}, \boldsymbol{n})\boldsymbol{\tau} \\
&= (\boldsymbol{I} - \boldsymbol{\Psi}(\boldsymbol{\gamma}^{(2)}, \boldsymbol{n})\boldsymbol{B})^{-1}\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(2)}, \boldsymbol{n})\boldsymbol{\tau} = \boldsymbol{p}^{(2)}
\end{aligned}$$

which proves the assertion using (9). ■

An immediate conclusion of Proposition 5 is that the above holds for the set $\mathcal{F}$, too.

Our results are visualized in Fig. 1 and Fig. 2. Fig. 1 shows the set $\mathcal{F}_{\mathcal{P}}$ for the two user case for $\gamma_i = 3.16,\ 10,\ 15$ when $\mathcal{P}$ is described by the ℓ_1-norm as $p_1 + p_2 \leq 10^{-4}$. According to Proposition 5 it can be seen that $\mathcal{F}_{\mathcal{P}}(15) \subset \mathcal{F}_{\mathcal{P}}(10) \subset \mathcal{F}_{\mathcal{P}}(3.16)$. Corresponding parameters are $G_{11} = G_{22} = 0.8, G_{12} = G_{21} = 0.2, \tau_b^2 = 1.9966 \cdot 10^{-3}, \eta_1 = \eta_2 = 10^{-11}, T_f = 10^{-7}$. The same parameters are used in Fig. 2, which depicts $\mathcal{F}_{\mathcal{P}}$ for $\gamma_i = 3.16, 10$ under the max-norm $\max\{p_1, p_2\} \leq 10^{-4}$. The curved lines show the extreme points with $p_2 = p_{\max}$ and variable $p_1 \in \{0, p_{\max}\}$, and vice versa.

The problem is now embedded into a more general framework, where the variables $\boldsymbol{n}$ may be arbitrary QoS parameters and ψ_i arbitrary log-convex functions. We aim at the same geometrical results for the set of feasible QoS parameters for impulse radio as known for CDMA systems. For this purpose we need a generalization of Theorem 2 in [6] as follows.

Proposition 6 *Suppose that ψ_i, $i = 1, \ldots, K$, are log-convex functions. Let $\boldsymbol{n}^{(0)}, \boldsymbol{n}^{(1)} \in \mathcal{F}$, $\boldsymbol{n}^{(0)} \neq \boldsymbol{n}^{(1)}$, and $\boldsymbol{n}^{(\lambda)} = \lambda\boldsymbol{n}^{(1)} + (1 - \lambda)\boldsymbol{n}^{(0)}$, $0 \leq \lambda \leq 1$. Then $p_i^*(\boldsymbol{n}^{(\lambda)})$ is a log-convex, and hence convex function of $\lambda \in [0, 1]$.*

Proof: The proof relies on the fact that the class $\mathcal{L}$ of log-convex functions on $[0, 1]$ augmented by the zero function is closed under the following operations, see [10], page 19. If $\alpha \geq 0$ and $f_1, f_2 \in \mathcal{L}$, then $\alpha f_1, f_1 + f_2, f_1 \cdot f_2 \in \mathcal{L}$. Moreover, if $f_j \in \mathcal{L}$, $j \in \mathbb{N}$, and $\sum_{j=1}^{\infty} f_j(\lambda) < \infty$ for $\lambda = 0, 1$, then $\sum_{j=1}^{\infty} f_j(\lambda) < \infty$ for all $\lambda \in [0, 1]$ and $\sum_{j=1}^{\infty} f_j \in \mathcal{L}$.

Now write

$$\begin{aligned}
\boldsymbol{p}^*(\boldsymbol{n}^{(\lambda)}) &= \left[\boldsymbol{I} - \boldsymbol{\Psi}(\boldsymbol{n}^{(\lambda)})\boldsymbol{B}\right]^{-1}\boldsymbol{\Psi}(\boldsymbol{n}^{(\lambda)})\boldsymbol{\tau} \\
&= \Big[\sum_{j=1}^{\infty}\left(\boldsymbol{\Psi}(\boldsymbol{n}^{(\lambda)})\boldsymbol{B}\right)^j\Big]\boldsymbol{\Psi}(\boldsymbol{n}^{(\lambda)})\boldsymbol{\tau}. \qquad (11)
\end{aligned}$$

By assumption $\boldsymbol{n}^{(0)}, \boldsymbol{n}^{(1)} \in \mathcal{F}$. From Proposition 1 it follows that $\rho\big(\boldsymbol{\Psi}(\boldsymbol{n}^{(j)}\big) < 1$, $j = 1, 2$, and the series (11) converges for $\lambda = 0$ and $\lambda = 1$.

Each entry of $\boldsymbol{\Psi}(\boldsymbol{n}^{(\lambda)}) = \operatorname{diag}\big(\psi_1(n_1^{(\lambda)}), \ldots, \psi_K(n_K^{(\lambda)})\big)$ is a log-convex function of $\lambda \in [0, 1]$. By the above and representation (11), $p_i^*(\boldsymbol{n}^{(\lambda)}) \in \mathcal{L}$ for all $i = 1, \ldots, K$, such that $p_i^*(\boldsymbol{n}^{(\lambda)})$ is a log-convex function of $\lambda \in [0, 1]$. The fact that log-convex functions are convex completes the proof. ∎

As a corollary to Proposition 6 the convexity of the set $\mathcal{F}_{\mathcal{P}}$ follows.

Proposition 7 *Suppose that ψ_i are log-convex functions, further that $\mathcal{P}$ is convex and satisfies* (9). *Then the constrained set $\mathcal{F}_{\mathcal{P}}$ is a convex set in $\mathbb{R}^K$.*

Proof: We start form two points $\boldsymbol{n}^{(0)}, \boldsymbol{n}^{(1)} \in \mathcal{F}_{\mathcal{P}}$. Hence, $\boldsymbol{p}^*(\boldsymbol{n}^{(0)}), \boldsymbol{p}^*(\boldsymbol{n}^{(1)}) \in \mathcal{P}$. Proposition 7 and the convexity of $\mathcal{P}$ yield

$$\boldsymbol{p}^*(\boldsymbol{n}^{(\lambda)}) \leq \lambda \boldsymbol{p}^*(\boldsymbol{n}^{(1)}) + (1-\lambda)\boldsymbol{p}^*(\boldsymbol{n}^{(0)}) \in \mathcal{P}$$

for all $0 \leq \lambda \leq 1$. By (9) it follows that $\boldsymbol{p}^*(\boldsymbol{n}^{(\lambda)}) \in \mathcal{P}$, and therefore $\boldsymbol{n}^{(\lambda)} \in \mathcal{F}_{\mathcal{P}}$ for all $0 \leq \lambda \leq 1$. ∎

Revisiting impulse radio with $\psi_i(n_i) = \frac{\gamma_i}{n_i T_f}$ Proposition 7 yields that the set of constrained fractional feasible repetition numbers is convex.

As an immediate consequence of Proposition 3 we furthermore obtain that $\mathcal{F}_{\mathcal{P}}$ is unbounded whenever the components of some feasible element are enlarged.

Proposition 8 *Suppose that the set $\mathcal{P}$ of admissible power allocations satisfies* (9). *If $\boldsymbol{n}^{(1)} \in \mathcal{F}_{\mathcal{P}}$ then $\boldsymbol{n} \in \mathcal{F}_{\mathcal{P}}$ for all $\boldsymbol{n} \geq \boldsymbol{n}^{(1)}$.*

V. Controlling Pulse Repetition Time

The pulse repetition time, T_f, may be used as a global system parameter to control proportional access of users in the case of overload. To see this, consider the entries of the matrix on the left hand side of (7). Its entries are

$$\begin{cases} 1 - \frac{\gamma_i}{n_i} \frac{\tau_b^2 G_{ij}}{G_i i}, & \text{if } i \neq j, \\ 1, & \text{if } i = j, \end{cases}$$

such that $\boldsymbol{I} - \boldsymbol{\Gamma R B}$ is independent of T_f. The ith entry of the right hand side of (7) reads as

$$\frac{1}{T_f} \frac{\gamma_i}{n_i} \frac{\eta_i}{G_{ii}}.$$

Now, if $\rho(\boldsymbol{\Gamma R B}) < 1$, then a power assignment $\boldsymbol{p}^* = \boldsymbol{p}^*(\boldsymbol{\gamma}, \boldsymbol{r})$ exists according to (8), however, it may happen that $\boldsymbol{p}^* \notin \mathcal{P}$. By Proposition 3, increasing T_f, i.e., enlarging the pulse repetition time, decreases $\boldsymbol{p}^*(\boldsymbol{\gamma}, \boldsymbol{r})$ componentwise with $\boldsymbol{p}^*(\boldsymbol{\gamma}, \boldsymbol{r}) \to 0$ as $T_f \to \infty$. Hence, if $\mathcal{P}$ is closed there exists

$$T_{f,\min} = \min\{T_f \mid \boldsymbol{p}^*(\boldsymbol{\gamma}, \boldsymbol{r}) \in \mathcal{P}\}.$$

$T_{f,\min}$ can be used as a global control parameter to achieve a power allocation

$$\boldsymbol{p}^* = [\boldsymbol{I} - \boldsymbol{\Gamma R B}]^{-1} \frac{1}{T_{f,\min}} \operatorname{diag}\big(\frac{\gamma_i}{n_i}\big)\, \boldsymbol{\tau}$$

with proportionally reduced QoS parameters $\frac{\gamma_i}{n_i}$, but $\boldsymbol{p}^* \in \mathcal{P}$.

VI. Conclusion

This paper has contributed to the interference and power control modeling of ultra-wideband impulse radio. Important parameters for controlling the system performance are power, quality-of-service demands in terms of the SIR, and pulse repetition time. The influence of each and their interaction has been investigated in the present work. It has been shown that both the set of admissible power allocations and the set of feasible reciprocal binary bit rates are convex and have certain monotonicity properties. After appropriate modeling of the key effects, analytical methods from CDMA have been extended and applied to the analysis of impulse radio. We have also suggested how to use the pulse repetition time for access control with proportionally reduced QoS parameters. Future research will be devoted to effective algorithms for determining the minimal reduction to achieve an admissible power adjustment.

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