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Tutorial 3

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Problem 1. *(Relative Entropy)*

Let the random variable X have three possible outcomes $\{a, b, c\}$. Consider two distributions on this random variable:

Symbol	p(x)	q(x)
a	$\frac{1}{2}$	$\frac{1}{3}$
b	$\frac{1}{4}$	$\frac{1}{3}$
c	$\frac{1}{4}$	$\frac{1}{3}$

a) Calculate $H(\mathbf{p})$, $H(\mathbf{q})$, $D(\mathbf{p}||\mathbf{q})$, and $D(\mathbf{q}||\mathbf{p})$. Verify in this case $D(\mathbf{p}||\mathbf{q}) \neq D(\mathbf{q}||\mathbf{p})$.

b) Although, $D(\mathbf{p}||\mathbf{q}) \neq D(\mathbf{q}||\mathbf{p})$ in general, there could be distributions for which equality holds. Give an example of two distributions $\mathbf{p}$ and $\mathbf{q}$ on a binary alphabet such that $D(\mathbf{p}||\mathbf{q}) = D(\mathbf{q}||\mathbf{p})$ (other than the trivial case $\mathbf{p} = \mathbf{q}$)).

Problem 2. *(Fano's Ineuality)*

Let $P(X = i) = p_i$, $i = 1, 2, ...m$, and let $p_1 \geq p_2 \geq p_3 \geq ... \geq p_m$. The minimal probability of error predictor of X is $\hat{X} = 1$ (there is no knowledge of Y), with resulting probability of error $P_e = 1 - p_1$.

a) Maximize $H(\mathbf{p})$ subject to the constraint $1 - p_1 = P_e$ to find a lower bound on P_e in terms of $H(\mathbf{p})$.

b) Find the probability vector $\mathbf{p} = (p_1, p_2, ...p_n)$ for which Fano's inequality is sharp i.e., $H(P_e) + P_e \log(m - 1) = H(\mathbf{p})$.

Problem 3. *(Bottle neck)*

Suppose that a non-stationary Markov chain starts in one of the n states, necks down to $k < n$ states, and then fans back to $m > k$ states. Thus, $X_1 \to X_2 \to X_3$, that is , $p(x_1, x_2, x_3) = p(x_1)p(x_2|x_1)p(x_3|x_2)$, for all $x_1 \in \{1, 2, ...n\}$, $x_2 \in \{1, 2, ...k\}$, $x_3 \in \{1, 2, ...m\}$.

a) Show that the dependence of X_1 and X_3 is limited by the bottle neck by proving that $I(X_1; X_3) \leq \log k$.

b) Evaluate $I(X_1; X_3)$ for $k = 1$, and conclude that no dependence can survive such a bottle neck.

Problem 4. Prove $\ln(t) \leq t - 1$ for $t \geq 0$