Information Theory

Chapter3: Source Coding

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Outline Chapter 2: Source Coding

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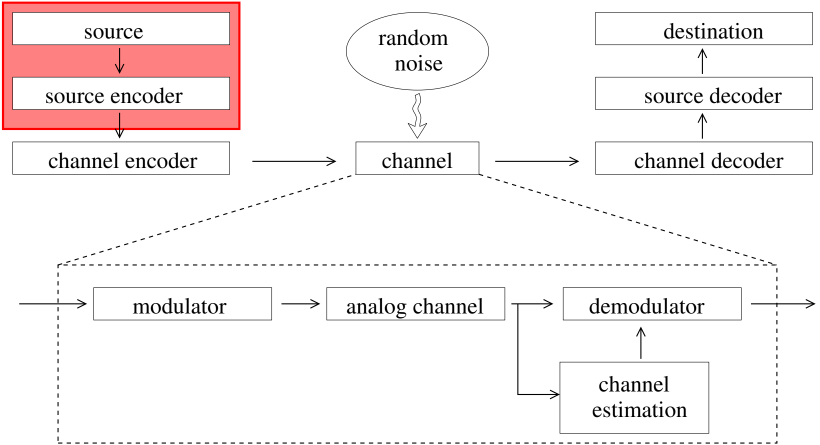
Arithmetic Coding

Communication Channel

from an information theoretic point of view



Variable Length Encoding

Given some
source alphabet $\mathcal{X} = \{x_1, \ldots, x_m\}$,
code alphabet $\mathcal{Y} = \{y_1, \ldots, y_d\}$.

Aim:
For each character $x_1, \ldots, x_m$ find a code word formed over $\mathcal{Y}$.

Formally:
Map each character $x_i \in \mathcal{X}$ uniquely onto a "word" over $\mathcal{Y}$.

Definition 3.1.
An injective mapping

$$g : \mathcal{X} \to \bigcup_{\ell=0}^{\infty} \mathcal{Y}^\ell : x_i \mapsto g(x_i) = (w_{i1}, \ldots, w_{in_i})$$

is called *encoding*. $g(x_i) = (w_{i1}, \ldots, w_{in_i})$ is called *code word* of character x_i, n_i is called *length* of code word i.

Variable Length Encoding

Example:

	g_1	g_2	g_3	g_4
a	1	1	0	0
b	0	10	10	01
c	1	100	110	10
d	00	1000	111	11
	no encoding	encoding, words are separable	encoding, shorter, words separable	encoding, even shorter, not separable

Hence, separability of concatenated words over $\mathcal{Y}$ is important.

Variable Length Encoding

Definition 3.2.
An encoding g is called *uniquely decodable (u.d.)* or *uniquely decipherable*, if the mapping

$$G : \bigcup_{\ell=0}^{\infty} \mathcal{X}^{\ell} \to \bigcup_{\ell=0}^{\infty} \mathcal{Y}^{\ell} : (a_1, \dots, a_k) \mapsto (g(a_1), \dots, g(a_k))$$

is injectiv.

Example:
Use the previous encoding g_3

	g_3
a	0
b	10
c	110
d	111

1 1 1 1 0 0 0 1 1 0 1 1 1 0 0 0 1 0
1 1 1|1 0 0 0 1 1 0 1 1 1 0 0 0 1 0
1 1 1|1 0 |0 0 1 1 0 1 1 1 0 0 0 1 0
1 1 1|1 0 |0|0 |1 1 0|1 1 1|0| 0|0|1 0
d b a a c d a a a b

(g_3 is a so called prefix code)

Prefix Codes

Definition 3.3.
A code is called *prefix code*, if no complete code word is prefix of some other code word, i.e., no code word evolves from continuing some other.

Formally:
$\boldsymbol{a} \in \mathcal{Y}^k$ is called prefix of $\boldsymbol{b} \in \mathcal{Y}^l$, $k \leq l$, if there is some $c \in \mathcal{Y}^{l-k}$ such that $\boldsymbol{b} = (\boldsymbol{a}, \boldsymbol{c})$.

Theorem 3.4.
Prefix codes are uniquely decodable.

More properties:

- Prefix codes are easy to construct based on the code word lengths.
- Decoding of prefix codes is fast and requires no memory storage.

Next aim: characterize uniquely decodable codes by their code word lengths.

Kraft-McMillan Theorem

Theorem 3.5. (a) McMillan (1959), b) Kraft (1949))

a) All uniquely decodable codes with code word lengths $n_1, \dots, n_m$ satisfy

$$\sum_{j=1}^{m} d^{-n_j} \leq 1$$

b) Conversely, if $n_1, \dots, n_m \in \mathbb{N}$ are such that $\sum_{j=1}^{m} d^{-n_j} \leq 1$, then there exists a u.d. code (even a prefix code) with code word lengths $n_1, \dots, n_m$.

Example:

	g_3	g_4
a	0	0
b	10	01
c	110	10
d	111	11
	u.d.	not u.d.

For g_3: $2^{-1} + 2^{-2} + 2^{-3} + 2^{-3} = 1$

For g_4:

$$2^{-1} + 2^{-2} + 2^{-2} + 2^{-2} = 5/4 > 1$$

g_4 is not u.d., there is no u.d. code with code word lengths 1,2,2,2.

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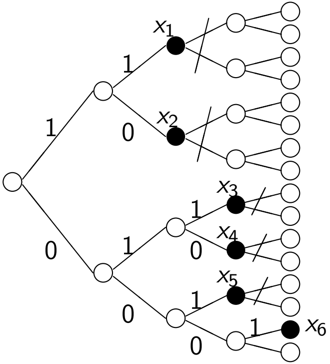
Kraft-McMillan Theorem, Proof of b)

Assume $n_1 = n_2 = 2,\ n_3 = n_4 = n_5 = 3,\ n_6 = 4.$
Then $\sum i = 1^6 = 15/16 < 1$
Construct a prefix code by a binary code tree as follows.



The corresponding code is given as

x_i	x_1	x_2	x_3	x_4	x_5	x_6
$g(x_i)$	11	10	011	010	001	0001

Average Code Word Length

Given a code $g(x_1), \dots, g(x_m)$ with code word lengths $n_1, \dots, n_m$.
Question: What is a reasonable measure of the "length of a code"?

Definition 3.6.
The *expected code word length* is defined as

$$\bar{n} = \bar{n}(g) = \sum_{j=1}^{m} n_j p_j = \sum_{j=1}^{m} n_j P(X = x_j)$$

Example:

	p_i	g_2	g_3
a	1/2	1	0
b	1/4	10	10
c	1/8	100	110
d	1/8	1000	111
$\bar{n}(g)$		15/8	14/8
$H(X)$	14/8		

Noiseless Coding Theorem, Shannon (1949)

Theorem 3.7.
Let random variable X describe a source with distribution $P(X = x_i) = p_i$, $i = 1, \dots, m$. Let the code alphabet $\mathcal{Y} = \{y_1, \dots, y_d\}$ have size d.

a) Each u.d. code g with code word lengths $n_1, \dots, n_m$ satisfies

$$\bar{n}(g) \geq H(X) / \log d.$$

b) Conversely, there is a prefix code, hence a u.d. code g with

$$\bar{n}(g) \leq H(X) / \log d + 1.$$

Proof of a)

For any u.d. code it holds by McMillan's Theorem that

$$
\begin{aligned}
\frac{H(X)}{\log d} - \bar{n}(g) &= \frac{1}{\log d} \sum_{j=1}^{m} p_j \log \frac{1}{p_j} - \sum_{j=1}^{m} p_j n_j \\
&= \frac{1}{\log d} \sum_{j=1}^{m} p_j \log \frac{1}{p_j} + \sum_{j=1}^{m} p_j \frac{\log d^{-n_j}}{\log d} \\
&= \frac{1}{\log d} \sum_{j=1}^{m} p_j \log \frac{d^{-n_j}}{p_j} \\
&= \frac{\log e}{\log d} \sum_{j=1}^{m} p_j \ln \frac{d^{-n_j}}{p_j} \\
&\leq \frac{\log e}{\log d} \sum_{j=1}^{m} p_j \left(\frac{d^{-n_j}}{p_j} - 1 \right) \\
&\leq \frac{\log e}{\log d} \sum_{j=1}^{m} \left(d^{-n_j} - p_j \right) \leq 0
\end{aligned}
$$

Proof of b) Shannon-Fano Coding

W.l.o.g. assume that $p_j > 0$ for all j.

Choose integers n_j such that $d^{-n_j} \leq p_j < d^{-n_j+1}$ for all j.
Then

$$\sum_{j=1}^{m} d^{-n_j} \leq \sum_{j=1}^{m} p_j \leq 1$$

such that by Kraft's Theorem a u.d. code g exists. Furthermore,

$$\log p_j < (-n_j + 1) \log d$$

holds by construction. Hence

$$\sum_{j=1}^{m} p_j \log p_j < (\log d) \sum_{j=1}^{m} p_j(-n_j + 1),$$

equivalently,

$$H(X) > (\log d) \left(\bar{n}(g) - 1\right).$$

Compact Codes

Is there always a u.d. code g with

$$\bar{n}(g) = H(X)/\log d?$$

No! Check the previous proof. Equality holds if and only if $p_j = 2^{-n_j}$ for all $j = 1, \ldots, m$.

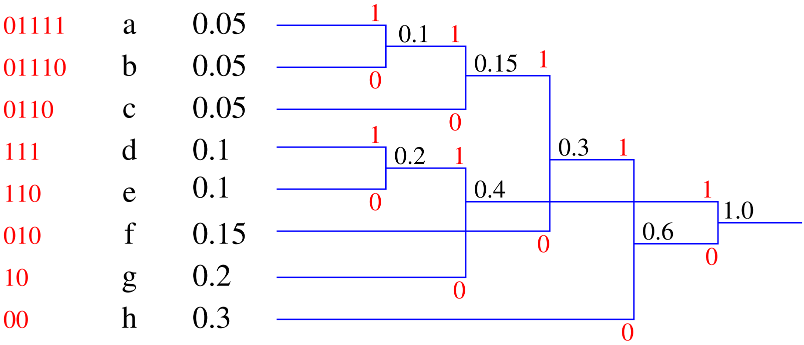
Example. Consider binary codes, i.e., $d = 2$. $\mathcal{X} = \{a, b\}$, $p_1 = 0.6,\ p_2 = 0.4$. The shortest possible code is $g(a) = (0),\ g(b) = (1)$.

$$\begin{aligned} H(X) &= -0.6 \log_2 0.6 - 0.4 \log_2 0.4 = 0.97095 \\ \bar{n}(g) &= 1. \end{aligned}$$

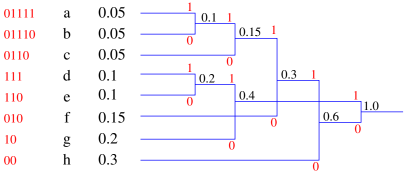
Definition 3.8.
Any code of shortest possible average code word length is called *compact*.

How to construct compact codes?

Huffman Coding



Huffman Coding



A compact code g^* is given by:

Character:	a	b	c	d	e	f	g	h
Code word:	01111	01110	0110	111	110	010	10	00

It holds (log to the base 2):

$$\bar{n}(g^*) = 5 \cdot 0.05 + \cdots + 2 \cdot 0.3 = \mathbf{2.75}$$
$$H(X) = -0.05 \cdot \log_2 0.05 - \cdots - 0.3 \cdot \log_2 0.3 = \mathbf{2.7087}$$

Block Codes for Stationary Sources

Encode blocks/words of length N by words over the code alphabet $\mathcal{Y}$.
Assume that blocks are generated by a stationary source, a stationary sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$.
Notation for a block code:

$$g^{(N)} : \mathcal{X}^N \to \bigcup_{\ell=0}^{\infty} \mathcal{Y}^\ell$$

Block codes are "normal" variabel length codes over the extended alphabet $\mathcal{X}^N$.

A fair measure of the "length" of a block code is the average code word length per character

$$\bar{n}\big(g^{(N)}\big)/N.$$

The lower Shannon bound, namely the entropy of the source, is asymptotically ($N \to \infty$) attained by suitable block codes, as is shown in the following.

Noiseless Coding Theorem for Block Codes

Theorem 3.9.
Let $\boldsymbol{X} = \{X_n\}_{n \in \mathbb{N}}$ be a stationary source. Let the code alphabet $\mathcal{Y} = \{y_1, \dots, y_d\}$ have size d.

a) Each u.d. block code $g^{(N)}$ satisfies

$$\frac{\bar{n}(g^{(N)})}{N} \geq \frac{H(X_1, \dots, X_N)}{N \, \log d}.$$

b) Conversely, there is a prefix block code, hence a u.d. block code $g^{(N)}$ with

$$\frac{\bar{n}(g^{(N)})}{N} \leq \frac{H(X_1, \dots, X_N)}{N \, \log d} + \frac{1}{N}.$$

Hence, in the limit as $N \to \infty$:
There is a sequence of u.d. block codes $g^{(N)}$ such that

$$\lim_{N \to \infty} \frac{\bar{n}(g^{(N)})}{N} = \frac{H_\infty(\boldsymbol{X})}{\log d}.$$

Huffman Block Coding

In principle, Huffman encoding can be applied to block codes. However, problems include

- The size of the Huffman table is m^N, thus growing exponentially with the block length.
- The code table needs to be transmitted to the receiver.
- The source statistics are assumed to be stationary. No adaptivity to to changing probabilities.
- Encoding and decoding only per block. Delays occur at the beginning and end. Padding may be necessary.

"Arithmetic coding" avoids these shortcomings.

Arithmetic Coding

Assume that

- Message $(x_{i_1}, \ldots, x_{i_N})$, $x_{i_j} \in \mathcal{X}$, $j = 1, \ldots, N$ is generated by some source $\{X_n\}_{n \in \mathbb{N}}$.
- All (conditional) probabilities

$$P(X_n = x_{i_n} \mid X_1 = x_{i_1}, \ldots, X_{n-1} = x_{i_{n-1}}) = p(i_n \mid i_1, \ldots, i_{n-1}),$$

$x_{i_1}, \ldots, x_{i_n} \in \mathcal{X}$, $n = 1, \ldots, N$, are known to the encoder and decoder, or can be estimated.

Then,

$$P(X_1 = x_{i_1}, \ldots, X_n = x_{i_n}) = p(i_1, \ldots, i_n)$$

can be easily computed as

$$p(i_1, \ldots, i_n) = p(i_n \mid i_1, \ldots, i_{n-1}) \cdot p(i_1, \ldots, i_{n-1})$$

Arithmetic Coding

Iterately construct intervals

Initialization, $n = 1$: $\big(c(1) = 0,\ c(m+1) = 1\big)$

$$I(j) = \big[c(j), c(j+1)\big), \quad c(j) = \sum_{i=1}^{j-1} p(i),\ j = 1, \dots, m$$

(cumulative probabilities)

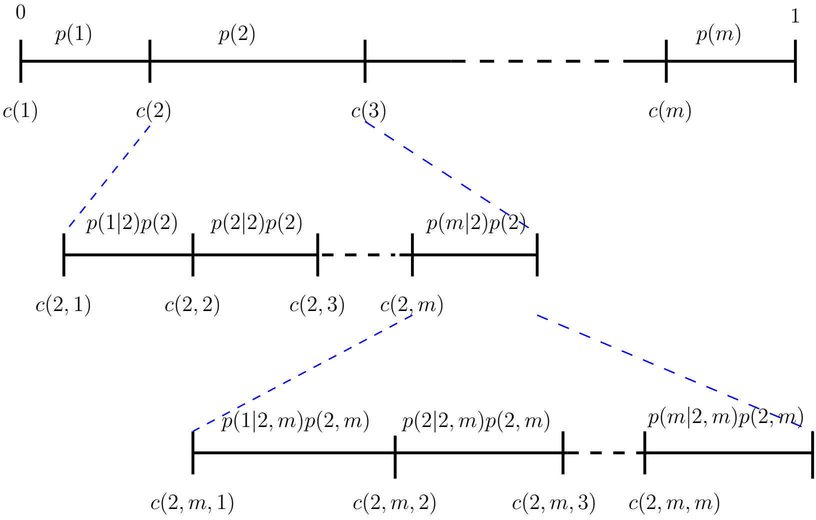
Recursion over $n = 2, \dots, N$:

$$\begin{aligned} &I(i_1, \dots, i_n) \\ &= \Big[c(i_1, \dots, i_{n-1}) + \sum_{i=1}^{i_n - 1} p(i_n \mid i_1, \dots, i_{n-1}) \cdot p(i_1, \dots, i_{n-1})\Big) \\ &\qquad c(i_1, \dots, i_{n-1}) + \sum_{i=1}^{i_n} p(i_n \mid i_1, \dots, i_{n-1}) \cdot p(i_1, \dots, i_{n-1})\Big) \end{aligned}$$

Program code available from Togneri, deSilva, p. 151, 152

Arithmetic Coding

Example.



Arithmetic Coding

Encode message $(x_{i_1}, \ldots, x_{i_N})$ by the binary representation of some binary number in the interval $I(i_1, \ldots, i_n)$.

A scheme which usually works quite well is as follows.
Let $l = l(i_1, \ldots, i_n)$ and $r = r(i_1, \ldots, i_n)$ denote the left and right bound of the corresponding interval. Carry out the binary expansion of l and r until until they differ. Since $l < r$, at the first place they differ there will be a 0 in the expansion of l and a 1 in the expansion of r. The number $0.a_1 a_2 \ldots a_{t-1} 1$ falls within the interval and requires the least number of bits.

$$(a_1 a_2 \ldots a_{t-1} 1) \text{ is the encoding of } (x_{i_1}, \ldots, x_{i_N}).$$

The probability of occurrence of message $(x_{i_1}, \ldots, x_{i_N})$ is equal to the length of the representing interval. Approximately

$$-\log_2 p(i_1, \ldots, i_n)$$

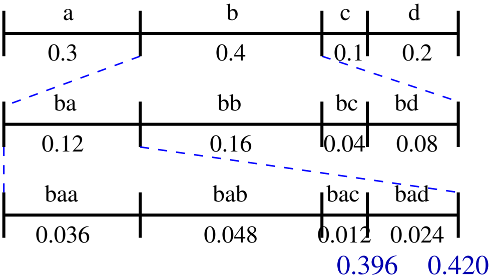
bits are needed to represent the interval, which is close to optimal.

Arithmetic Coding

Example. Assume a memoryless source with 4 characters and probabilities

x_i	a	b	c	d
$P(X_n = x_i)$	0.3	0.4	0.1	0.2

Encode the word (bad):



$$(bad) = [0.396, 0.42)$$

$$0.396 = 0.01100\ldots \qquad 0.420 = 0.01101\ldots$$

$$(bad) = (01101)$$