Wireless Channel Modeling and Propagation Effects

Rudolf Mathar

RWTH Aachen University, November 2009

Outline

Log-normal Fading

Well established model for distance dependent average power attenuation:

$$P_r(d) = P_r(d_0)\left(\frac{d}{d_0}\right)^{-\gamma}, \quad 2 \le \gamma \le 5,$$

d_0 reference distance.
Equivalently, path loss in dB

$$L(d) = L(d_0) + 10\,\gamma\,\log\frac{d}{d_0}$$

Table of typical values:

Propagation environment	γ
Free space	2
Ground-wave reflection	4
Urban cellular radio	2.7 - 3.5
Shadowed cellular radio	3 - 5
In-building line-of-sight	1.6 - 1.8
Obstructed in-building	4 - 6

Log-normal Fading

Additional multiplicative random effects:

$$P_r(d) = P_r(d_0)\left(\frac{d}{d_0}\right)^{-\gamma}\prod_{i=1}^{N} X_i.$$

Equivalently, for the path loss in dB

$$L(d) = L(d_0) + 10\,\gamma\,\log\frac{d}{d_0} + 10\sum_{i=1}^{N}\log X_i$$

Gaussian approximation, $X = 10\sum_{i=1}^{N}\log X_i \sim N(0,\sigma^2)$:

$$L(d) = L(d_0) + 10\,\gamma\,\log\frac{d}{d_0} + X \quad (\text{dB})$$

with

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\,e^{-\frac{x^2}{2\sigma^2}}$$

σ^2 measured in dB. From practical measurement $\sigma^2 \in [4, 12]$, typically $\sigma^2 = 8dB$.

Log-normal Fading

Set the multiplicative random fading

$$Y = \prod_{i=1}^{N} X_i = 10^{X/10}$$

If $X \sim N(0, \sigma^2)$, the pdf of Y is

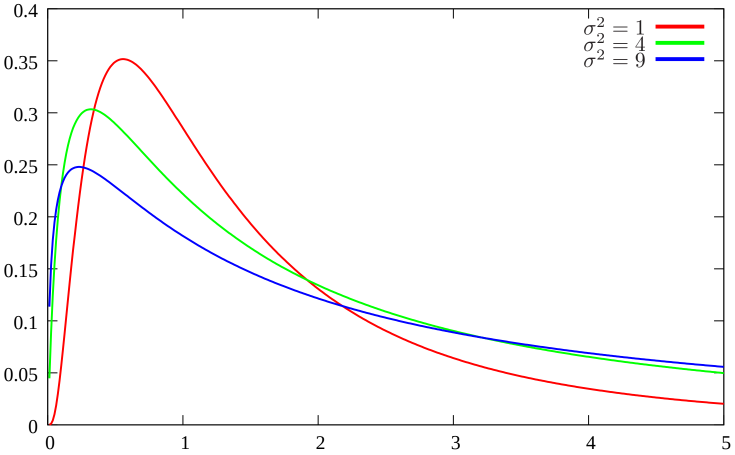
$$f_Y(y) = \frac{10}{\ln 10 \cdot \sqrt{2\pi}\,\sigma y} \exp\left(- \frac{(10 \log y)^2}{2\sigma^2} \right), \quad y \geq 0.$$

- The distribution of Y is called *log-normal distribution*.
- Hence, Y is log-normally distributed since $\log Y$ is normallly distributed.
- A more general form: Let $X \sim N(\mu, \sigma^2)$, $Y = e^X$. Then

$$f_Y(y) = \frac{1}{y\sqrt{2\pi}\,\sigma} \exp\left(- \frac{(\ln y - \mu)^2}{2\sigma^2} \right), \; y > 0.$$

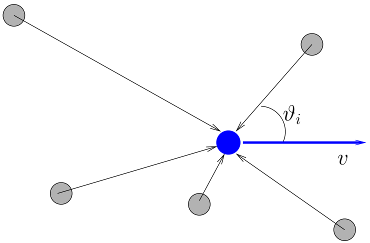
Demonstrated on whiteboard.

Log-normal Fading



Densities of the log-normal distribution for $\sigma^2 \in \{1, 4, 9\}$.

Scattering Model



Doppler shift for scatterer i: $D_i = +\frac{f}{c} v \cos\theta_i$

No direct line of sight, only reflected signals are received.

Total received signal for n scatterers/reflectors of an unmodulated signal $s(t) = e^{j2\pi ft}$:

$$r(t) = \sum_{i=1}^{n} A_i e^{j\left[2\pi f(t+\frac{vt}{c}\cos\theta_i)+\Phi_i\right]}$$

A_i: random amplitudes Φ_i: random phase shifts

Scattering Model (ctd)

Total received signal for n scatterers/reflectors:

$$r(t) = \sum_{i=1}^{n} A_i e^{j\left[2\pi f(t+\frac{vt}{c}\cos\theta_i)+\Phi_i\right]}$$

Assumptions:

$\Phi_i \sim \mathrm{R}[0,2\pi]$	Random phase shifts due to reflection and path length, uniformly distributed over $[0,2\pi]$.
A_i	Random amplitudes, identically distributed random variables
$E(A_i^2) = \frac{\sigma^2}{n}$	implies $\sum_i E(A_i^2) = \sigma^2$ (average received power)
$A_1,\dots,A_n$, $\Phi_1,\dots,\Phi_n$	jointly stochastically independent

Scattering Model (ctd)

With

$$c_i = 2\pi f \frac{v}{c} \cos\theta_i$$

write the received signal as

$$\begin{aligned} r(t) &= e^{j2\pi ft} \sum_{i=1}^{n} A_i e^{j\left[c_i t + \Phi_i\right]} \\ &= e^{j2\pi ft} \Big(\underbrace{\sum_{i=1}^{n} A_i \cos(c_i t + \Phi_i)}_{X(t)} + j \underbrace{\sum_{i=1}^{n} A_i \sin(c_i t + \Phi_i)}_{Y(t)} \Big) \\ &= e^{j2\pi ft} \big(X(t) + j\, Y(t) \big) \end{aligned}$$

Scattering Model (ctd)

Fix t in $X(t)$ and $Y(t)$.

Facts

- $\cos(c_i t + \Phi_i)$ and $\cos(\Phi_i)$ have the same distribution, likewise
- $\sin(c_i t + \Phi_i)$ and $\sin(\Phi_i)$ have the same distribution,
- $\mathrm{E}(\cos \Phi_i) = \mathrm{E}(\sin \Phi_i) = 0$

Hence

$$\mathrm{E}\left(\sqrt{n} A_i \cos(c_i t + \Phi_i)\right) = 0$$

$$\mathrm{E}\left(n A_i^2 \cos^2(c_i t + \Phi_i)\right) = \sigma^2\, \mathrm{E}(\cos^2(\Phi)) = \frac{\sigma^2}{2} \quad \text{and}$$

$$\mathrm{Var}\left(\sqrt{n} A_i \cos(c_i t + \Phi_i)\right) = \frac{\sigma^2}{2}$$

By the Central Limit Theorem (CLT)

$$X(t) = \sum_{i=1}^{n} A_i \cos(c_i t + \Phi_i) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sqrt{n} A_i \cos(c_i t + \Phi_i) \overset{\text{as}}{\sim} N\Big(0, \frac{\sigma^2}{2}\Big)$$

Scattering Model (ctd)

Analogously, the same holds for $Y(t)$. Hence

$$X(t) \stackrel{\text{as}}{\sim} N\left(0, \frac{\sigma^2}{2}\right) \quad \text{and} \quad Y(t) \stackrel{\text{as}}{\sim} N\left(0, \frac{\sigma^2}{2}\right)$$

Moreover, $X(t)$ and $Y(t)$ are uncorrelated, since

$$\begin{aligned} &\mathrm{E}\left[\Big(\sum_i A_i \cos(c_i t + \Phi_i)\Big)\Big(\sum_k A_k \sin(c_k t + \Phi_k)\Big)\right] \\ &= \sum_{i,k} \mathrm{E}\left[A_i A_k \cos(c_i t + \Phi_i) \sin(c_k t + \Phi_k)\right] \\ &= \sum_i \mathrm{E}\Big[A_i^2 \underbrace{\cos(c_i t + \Phi_i) \sin(c_i t + \Phi_i)}_{=\frac{1}{2}\sin(2(c_i t + \Phi_i))}\Big] \\ &= \sum_i \frac{\sigma^2}{2n}\ \mathrm{E}\left[\sin(2(c_i t + \Phi_i))\right] = 0 \end{aligned}$$

Rayleigh Distribution

In summary,

$$r(t) = e^{j2\pi ft}\big(X(t) + jY(t)\big)$$

with $X(t), Y(t)$ i.i.d. $\sim N(0, \frac{\sigma^2}{2})$.

The signal at time t is hence

- randomly attenuated by

$$R = \sqrt{X(t)^2 + Y(t)^2}$$

- randomly shifted in phase by

$$\Phi = \angle\{X(t) + jY(t)\}.$$

Problem: What is the joint distribution of R and Φ?

Interlude: Transformation of Random Vectors

Let $\mathbf{X} \in \mathcal{R}^n$ be a random vector with density $f_{\mathbf{X}}(\mathbf{x})$ such that $f_{\mathbf{X}}(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{M}$, $\mathcal{M} \subseteq \mathcal{R}^n$ an open set.

$T : \mathcal{R}^n \to \mathcal{R}^n$ an injective transformation such that

$$J(\mathbf{x}) = \left| \left(\frac{\partial T_i}{\partial x_j} \right)_{1 \le i,j \le n} \right| > 0 \text{ for all } \mathbf{x} \in \mathcal{M}.$$

Then $\mathbf{Y} = T(\mathbf{X})$ has a density

$$\begin{aligned} f_{\mathbf{Y}}(\mathbf{y}) &= \frac{1}{\left| J(\mathbf{x}) \big|_{T^{-1}(\mathbf{y})} \right|} \, f_{\mathbf{X}}\big(T^{-1}(\mathbf{y})\big) \\ &= \left| \tilde{J}(\mathbf{y}) \right| \, f_{\mathbf{X}}\big(T^{-1}(\mathbf{y})\big), \qquad \mathbf{y} \in T(\mathcal{M}), \end{aligned}$$

where $\tilde{J}(\mathbf{y}) = \left(\frac{\partial T_i^{-1}}{\partial y_j} \right)_{1 \le i,j \le n}$.

Rayleigh Distribution (ctd)

Back to $(X(t) + jY(t))$, suppress t, set $\tau^2 = \sigma^2/2$.

Joint density

$$f_{(X,Y)}(x,y) = \frac{1}{\sqrt{2\pi}\tau} e^{-\frac{x^2}{2\tau^2}} \frac{1}{\sqrt{2\pi}\tau} e^{-\frac{y^2}{2\tau^2}}$$

Transformation to polar coordinates:

$$(r,\varphi) = T(x,y), \text{ with } r = \sqrt{x^2+y^2}, \varphi = \angle(x,y)$$

Inverse transformation:

$$T^{-1}(r,\varphi) = (r\cos\varphi, r\sin\varphi), \quad r > 0,\ 0 < \varphi \le 2\pi$$

Jacobian of the inverse:

$$\left|\tilde{J}(r,\varphi)\right| = |r|$$

Rayleigh Distribution (ctd)

By the density transformation theorem:

$$f_{(R,\Phi)}(r,\varphi) = r\,\frac{1}{2\pi\tau^2}\,e^{-\frac{r^2\cos^2\varphi + r^2\sin^2\varphi}{2\tau^2}}, \quad 0 < r,\ 0 < \varphi \le 2\pi$$

$$= \underbrace{\frac{r}{\tau^2}\,e^{-\frac{r^2}{2\tau^2}}\,\mathbb{I}_{(0,\infty)}(r)}_{\sim Ray(\tau^2)} \cdot \underbrace{\frac{1}{2\pi}\,\mathbb{I}_{(0,2\pi]}(\varphi)}_{\sim U(0,2\pi)}$$

Hence, in

$$r(t) = e^{j2\pi ft}\big(X(t) + jY(t)\big)$$

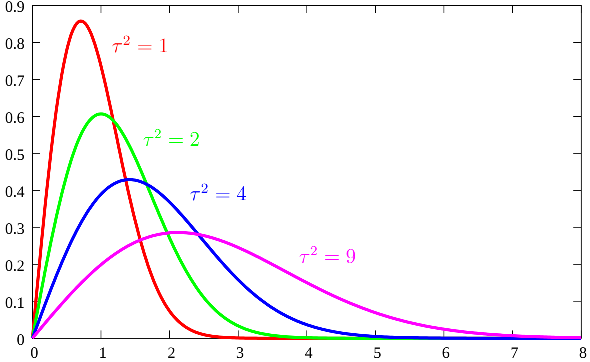
the amplitude $R(t)$ and phase $\Phi(t)$ of $\big(X(t) + jY(t)\big)$ are stochastically independent random variables with densities

$$f_R(r) = \frac{r}{\tau^2}\,e^{-\frac{r^2}{2\tau^2}}, \quad r > 0 \qquad \text{(Rayleigh distribution)}$$

$$f_\Phi(\varphi) = \frac{1}{2\pi}, \quad 0 < \varphi \le 2\pi \qquad \text{(uniform distribution)}$$

Rayleigh Distribution (ctd)

Plot of different Rayleigh densities



$$f(r) = \frac{2r}{\tau^2}\, e^{-r^2/\tau^2}, \quad \tau^2 = 1, 2, 4, 9$$

Rayleigh Distribution (ctd)

Note that

$$Z = R^2 \text{ with } R \sim \text{Ray}(\tau^2)$$

is exponentially distributed with density

$$f_Z(z) = \frac{1}{2\tau^2}\, e^{-z/2\tau^2}, \quad z > 0$$

Hence, the instantaneous power $Z = R^2$

$$R^2 = |X + jY|^2 = X^2 + Y^2$$

of a Rayleigh fading signal is exponentially distributed with parameter $\frac{1}{2\tau^2} = \frac{1}{\sigma^2}$, σ^2 being the expected receive power.

Rayleigh Fading Process

Recall the fading process over time $t \in \mathcal{R}$:

$$r(t) = e^{j2\pi ft}\Big(\underbrace{\sum_{i=1}^{n} A_i \cos(c_i t + \Phi_i)}_{X(t)} + j\underbrace{\sum_{i=1}^{n} A_i \sin(c_i t + \Phi_i)}_{Y(t)}\Big)$$

with $c_i = 2\pi f \frac{v}{c} \cos\theta_i$. From the above

$$\mathrm{E}\left(X(t)\right) = \mathrm{E}\left(Y(t)\right) = 0 \quad \text{for all } t$$

$$\mathrm{E}\left(X^2(t)\right) = \mathrm{E}\left(Y^2(t)\right) = \frac{\sigma^2}{2} \quad \text{for all } t$$

$$\mathrm{Cov}\big(X(t_1), Y(t_2)\big) = 0 \quad \text{for all } t_1, t_2$$

Define the *autocorrelation function* of $X(t)$

$$R_{XX}(\tau) = \mathrm{E}\left(X(t)X(t+\tau)\right) = \mathrm{Cov}\big(X(t), X(t+\tau)\big)$$

Rayleigh Fading Process (ctd.)

Autocorrelation function:

$$
\begin{aligned}
R_{XX}(\tau) &= \mathrm{E}\left((X(t)\,X(t+\tau)\right) \\
&= \mathrm{E}\left(\sum_{i,k} A_i A_k \cos(c_i t + \Phi_i)\cos(c_k(t+\tau)+\Phi_k)\right) \\
&= \mathrm{E}\left(\sum_{i} A_i^2 \cos(c_i t + \Phi_i)\cos(c_i(t+\tau)+\Phi_i)\right) \\
&= \frac{1}{2}\sum_i \mathrm{E}\left(A_i^2\right)\mathrm{E}\left(\cos(c_i\tau) + \cos(2c_i t + c_i\tau + 2\Phi_i)\right) \\
&= \frac{\sigma^2}{2n}\sum_i \cos\left(2\pi f \frac{v}{c}\tau\cos\theta_i\right)
\end{aligned}
$$

where we have used $\cos\alpha\cos\beta = \frac{1}{2}[cos(\alpha-\beta)+\cos(\alpha+\beta)]$.

Rayleigh Fading Process (ctd.)

Assume furthermore that $\theta_i \sim \mathrm{R}(0, 2\pi)$ is stochastically independent of A_i and Φ_i, and uniformly distributed over $[0, 2\pi]$. Then

$$\begin{aligned} R_{XX}(\tau) &= \frac{\sigma^2}{2}\frac{1}{2\pi}\int_0^{2\pi} \cos\left(2\pi f \frac{v}{c}\tau \cos\theta\right) d\theta \\ &= \frac{\sigma^2}{2}\frac{1}{\pi}\int_0^{\pi} \cos\left(2\pi f \frac{v}{c}\tau \cos\theta\right) d\theta \\ &= \frac{\sigma^2}{2}\mathrm{Re}\big(J_0(2\pi f \frac{v}{c}\tau)\big) \\ &= \frac{\sigma^2}{2}\mathrm{Re}\big(J_0(2\pi \frac{v}{\lambda}\tau)\big) = \frac{\sigma^2}{2}\mathrm{Re}\big(J_0(2\pi f_D \tau)\big) \end{aligned}$$

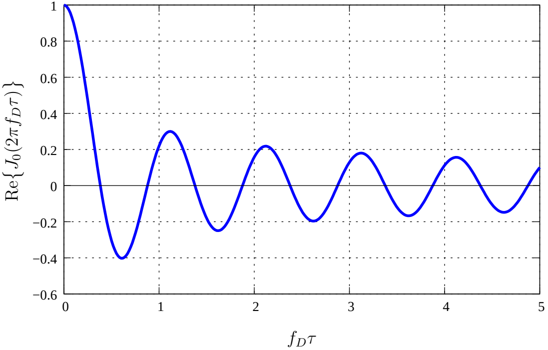
where $f_D = v/\lambda$ the maximum Doppler shift and

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} e^{-jx\cos\theta} d\theta$$

denotes the zeroth order Bessel function of the first kind.

Rayleigh Fading Process (ctd.)

Plot of $\mathrm{Re}\{J_0(2\pi f_D \tau)\}$ as a function of $f_D\tau$:



We see that

$$R_{XX}(\tau) = 0, \text{ if } f_D\tau \approx 0.4.$$

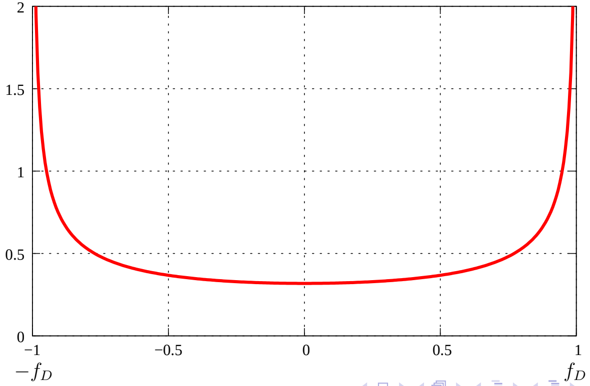
Conclusion: the signal decorrelates if $v\tau = 0.4\lambda =$ approximately a distance of one half wavelength.

Rayleigh Fading Process (ctd.)

The power spectral density of $X(t)$ is given by

$$\mathcal{F}(R_{XX})(f) = \begin{cases} \frac{\sigma^2}{\pi f_D} \frac{1}{\sqrt{1-(f/f_D)^2}}, & \text{if } |f| \leq f_D \\ 0, & \text{otherwise} \end{cases}$$

Graph of $\mathcal{F}(R_{XX})(f)$ for $f_D = 1,\ \sigma^2 = 1$:



Rayleigh Fading Process (ctd.)

Remark: Exactly the same goes through for the imaginary part $Y(t)$ of

$$r(t) = e^{j2\pi ft}\big(X(t) + j\,Y(t)\big),$$

so

$$R_{YY}(\tau) = \frac{\sigma^2}{2}\mathrm{Re}\big(J_0(2\pi f_D \tau)\big)$$

and

$$\mathcal{F}\big(R_{YY}\big)(f) = \begin{cases} \frac{\sigma^2}{\pi f_D}\ \frac{1}{\sqrt{1-(f/f_D)^2}}, & \text{if } |f| \le f_D \\ 0, & \text{otherwise} \end{cases}.$$

Furthermore, the processes $\{X(t)\}$ and $\{Y(t)\}$ are uncorrelated.

Rice Distribution

Recall:

$$X, Y \text{ i.i.d.} \sim N(0, \tau^2) \Longrightarrow \sqrt{X^2 + Y^2} \sim \text{Ray}(\tau^2)$$

This models the case with no LOS.

If additionally there is a LOS path, then

X, Y stochastically independent, $X \sim N(\mu_1, \tau^2)$, $Y \sim N(\mu_2, \tau^2)$.

In this case, $R = \sqrt{X^2 + Y^2}$ is *Rician* distributed with density

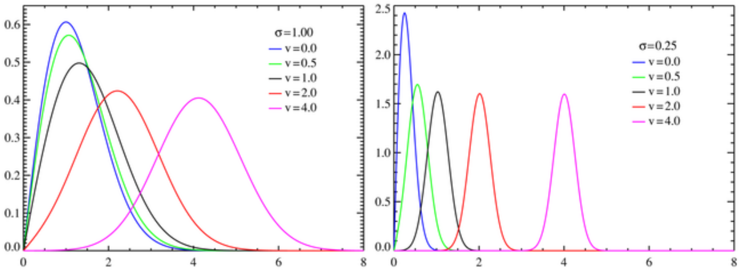
$$f_R(r) = \frac{r}{\tau^2} \exp\Big(-\frac{r^2 + \mu^2}{2\tau^2}\Big) I_0\Big(\frac{r\mu}{\tau^2}\Big), \quad r > 0,$$

where

$$\mu = \sqrt{\mu_1^2 + \mu_2^2}, \quad \text{and} \quad I_0(x) = \frac{1}{\pi} \int_0^\pi e^{x \cos \vartheta} d\vartheta$$

denotes the modified Bessel function of zeroth order.

Rice Distribution



Rician densities (from Wikipedia) ($\sigma \mathrel{\hat{=}} \tau$, $v \mathrel{\hat{=}} \mu$). Note that $v = \mu = 0$ corresponds to Rayleigh fading.