The Capacity Region of 3G CDMA Systems with Macrodiversity and Power Constraints

Virgilio Rodriguez
Polytechnic University
ECE Department
5 Metrotech Center
Brooklyn, NY 11201
Email: vr@ieee.org

***Abstract*— Under macro-diversity, the cellular structure is removed and each transmitter is jointly decoded by all "receivers" (base stations, or antennas in a single cell). This scheme has been shown to increase the capacity of CDMA wireless networks. However, the available macrodiversity capacity results rely on a "self-interference" approximation, which may *not* be appropriate in CDMA system in which the spreading gain is low for some transmitters. This is expected in 3rd generation cellular systems. Explicitly considering power constraints, and without recurring to the above approximation, this note applies well established "fixed point" theorems to derive less conservative results defining the macrodiversity capacity region.**

I. Introduction

As described by Hanly [2], macrodiversity is a scheme in which the cellular structure of a wireless communication network is removed and "each mobile...(is)...jointly decoded by all receivers in the network". Alternatively, one can think of a single-cell network equipped with several receiving antennas, possibly distributed in various locations throughout the cell. Hanly [2] shows that this scheme can significantly increase the capacity of a CDMA wireless communication network.

The macrodiversity capacity results provided by [2] assume that the transmission power of each transmitter contributes to its own interference. This approximation is generally appropriate for a CDMA system in which each transmitter's spreading gain is "large", which, normally means that its (pre-spread) "carrier to interference ratio" is "small".

But modern wireless networks are expected to accommodate simultaneous transceivers operating at a wide range of data rates. "Variable spreading gain" (VSG) CDMA is one of the technologies through which new standards accommodate such multi-rate traffic (see for instance, Nanda, et al.[6]). In a VSG-CDMA system (see I and Sabnani[3]), each transceiver's spreading gain is determined as the ratio of the common chip rate to the transceiver's data rate. Thus, high data rate sources generally operate with "low" spreading gains, and "high" carrier-to-interference ratios. Under these conditions, the "self-interference" approximation may not be appropriate.

Explicitly considering transmission power limits, and without recurring to the "self-interference" approximation, this note derives results determining the capacity region of a CDMA cellular network under macrodiversity. The "complexity" of applying the new results is comparable to that of the approximated ones. The analysis is grounded on the Brouwer's fixed point theorem and the Banach's contraction mapping principle, two well established mathematical results.

Below, the basic macrodiversity relation is presented, first in the traditional form, and subsequently in matrix form, in terms of convenient new variables. Then, it is shown that the basic macrodiversity capacity question is equivalent to determining whether certain meaningful function has a fixed point. Subsequently, conditions are identified under which the desired solution exists. Moreover, further conditions are explored under which this solution is unique, and can be determined through an intuitive, well-behaved algorithm. Finally, the results are interpreted and discussed. Space limitations preclude a comprehensive comparison between the new results and those previously available. Nevertheless, some brief contrasting comments are made, highlighting the fact that the new results are less conservative, which can make a significant difference in the throughput of a 3G system.

A mathematical appendix introduces the essential mathematical terminology, and some technical results.

II. The macro-diversity framework

A. Basic relation

Under macro-diversity, the cellular structure is removed and each transmitter is jointly decoded by all "receivers" (base stations, or antennas in a single cell). Hanly [2] argues that, in this situation, a relevant QoS index for terminal i is the product of its spreading gain by α_i, defined as:

$$\alpha_i = \frac{P_i h_{i1}}{\sum_{\substack{j=1 \\ j\neq i}}^{N} P_j h_{j1} + \sigma_1^2} + \cdots + \frac{P_i h_{iK}}{\sum_{\substack{j=1 \\ j\neq i}}^{N} P_j h_{jK} + \sigma_K^2} \quad (1)$$

K is the number of "receivers" in the network, and h_{ik} is the "path loss" coefficient in the signal from terminal i when received at k. α_i can be thought of as a desired "carrier to interference ratio" (CIR).

B. The Capacity question

Conditions are sought under which a given N-vector of positive numbers, $\vec{\alpha} := \begin{bmatrix} \alpha_1 & \cdots & \alpha_N \end{bmatrix}^{\mathrm{t}}$, is such that there exists another N-vector of positive numbers, $\begin{bmatrix} P_1 & \cdots & P_N \end{bmatrix}^{\mathrm{t}}$, satisfying appropriate constraints, and equation (1) for each i. If this is the case, the system of N equations like (1) has a feasible solution, and the vector of power ratios $\vec{\alpha}$ is said to be in the "capacity region" of the system.

C. Normalizations and re-formulations

Noise normalization. Let all powers be divided by $\sigma_1^2 + \cdots + \sigma_K^2$. Also, let $\nu_k = \sigma_k^2/(\sigma_1^2 + \cdots + \sigma_K^2)$. Although this normalization introduces no notational change on the power vector, it is understood that henceforth all powers are expressed as multiple of the total noise power $\sigma_1^2 + \cdots + \sigma_K^2$.

Total received power from a given transmitter. Let

$$Q_i := P_i \sum_{k=1}^{K} h_{ik} \tag{2}$$

Scaled power. Let $q_i := Q_i/\alpha_i$ (The total received power from terminal i is "scaled" by that terminal's desired CIR α_i).

Relatives losses. Let

$$g_{ik} := \frac{h_{ik}}{\sum_{j=1}^{K} h_{ij}} \tag{3}$$

The power at receiver k coming from transmitter i, $P_i h_{ik} = g_{ik}\alpha_i q_i$.

Now, the basic macro-diversity equation can be restated as:

$$\frac{q_i g_{i1}}{\sum_{\substack{j=1\\ j\neq i}}^{N} \alpha_j q_j g_{j1} + \nu_1} + \cdots + \frac{q_i g_{iK}}{\sum_{\substack{j=1\\ j\neq i}}^{N} \alpha_j q_j g_{jK} + \nu_K} = 1 \tag{4}$$

Notice that $P_i = \alpha_i q_i / \sum_{k=1}^{K} h_{ik}$, which is measured as a multiple of the total noise power $\sigma_1^2 + \cdots + \sigma_K^2$.

D. Macrodiversity matrix relations

Let

$$Y_{ik}(\vec{q}) := \sum_{\substack{j=1\\ j\neq i}}^{N} \alpha_j q_j g_{jk} + \nu_k \tag{5}$$

$Y_{ik}(\vec{q})$ can be written as the scalar product of vectors as:

$$\begin{bmatrix} g_{1k} & \cdots & g_{i-1,k} & 0 & g_{i+1,k} & \cdots & g_{Nk} \end{bmatrix} \cdot D\vec{q} + \nu_k$$

with

$$D := \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \alpha_N \end{bmatrix} \tag{6}$$

so that,

$$D\vec{q} = \begin{bmatrix} \alpha_1 q_1 \\ \alpha_2 q_2 \\ \vdots \\ \alpha_N q_N \end{bmatrix}$$

It will prove useful to recognize the vectors $\vec{Y}_i(\vec{q}) := \begin{bmatrix} Y_{i1} & \cdots & Y_{iK} \end{bmatrix}^{\mathrm{t}}$.

By "stacking" these interference vectors, one arrives at a "macro-vector" of length NK satisfying:

$$\vec{Y} := \begin{bmatrix} \vec{Y}_1 \\ \vec{Y}_2 \\ \vdots \\ \vec{Y}_{N-1} \\ \vec{Y}_N \end{bmatrix} = \mathcal{G}D \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{N-1} \\ q_N \end{bmatrix} + \begin{bmatrix} \vec{\nu} \\ \vec{\nu} \\ \vdots \\ \vec{\nu} \\ \vec{\nu} \end{bmatrix} \tag{7}$$

where $\mathcal{G}$ is a matrix defined as

$$\begin{bmatrix} \vec{0} & \vec{g}_2 & \cdots & \vec{g}_{N-1} & \vec{g}_N \\ \vec{g}_1 & \vec{0} & \cdots & \vec{g}_{N-1} & \vec{g}_N \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \vec{g}_1 & \vec{g}_2 & \cdots & \vec{0} & \vec{g}_N \\ \vec{g}_1 & \vec{g}_2 & \cdots & \vec{g}_{N-1} & \vec{0} \end{bmatrix} \equiv \begin{bmatrix} \mathcal{G}^1 \\ \mathcal{G}^2 \\ \vdots \\ \mathcal{G}^{N-1} \\ \mathcal{G}^N \end{bmatrix} \tag{8}$$

with $\vec{0}$ the zero vector of appropriate length, and

$$\vec{g}_i := \begin{bmatrix} g_{i1} \\ \vdots \\ g_{iK} \end{bmatrix} \quad \vec{\nu} := \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_K \end{bmatrix} \quad \vec{\boldsymbol{\nu}} := \begin{bmatrix} \vec{\nu} \\ \vdots \\ \vec{\nu} \end{bmatrix} \tag{9}$$

The matrix $\mathcal{G}D$ is some times denoted as $\hat{\mathcal{G}}$. $\mathcal{G}^{ik}$ (respect. $\hat{\mathcal{G}}^{ik}$) may denote the specific *row* of $\mathcal{G}$ (respect. $\hat{\mathcal{G}}$) "matching" Y_{ik}, with $\mathcal{G}^i$ (respect. $\hat{\mathcal{G}}^i$) the corresponding sub-matrix. Thus,

$$Y_{ik} = \mathcal{G}^{ik} \cdot D \cdot \vec{q} + \nu_k \equiv \hat{\mathcal{G}}^{ik} \cdot \vec{q} + \nu_k \tag{10}$$

The preceding notational transformations can be clarified by considering the specific case in which there are N=3 transmitters and K=2 receivers. In this case:

$$\vec{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \;;\; D = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}$$

$$\vec{Y}_1 \equiv \begin{bmatrix} Y_{11} \\ Y_{12} \end{bmatrix} = \begin{bmatrix} 0 & g_{21} & g_{31} \\ 0 & g_{22} & g_{32} \end{bmatrix} \begin{bmatrix} \alpha_1 q_1 \\ \alpha_2 q_2 \\ \alpha_3 q_3 \end{bmatrix} + \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \equiv \begin{bmatrix} \vec{0} & \vec{g}_2 & \vec{g}_3 \end{bmatrix} D\vec{q} + \vec{\nu}$$

$$\vec{Y}_2 \equiv \begin{bmatrix} Y_{21} \\ Y_{22} \end{bmatrix} = \begin{bmatrix} g_{11} & 0 & g_{31} \\ g_{12} & 0 & g_{32} \end{bmatrix} \begin{bmatrix} \alpha_1 q_1 \\ \alpha_2 q_2 \\ \alpha_3 q_3 \end{bmatrix} + \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \equiv \begin{bmatrix} \vec{g}_1 & \vec{0} & \vec{g}_3 \end{bmatrix} D\vec{q} + \vec{\nu}$$

$$\vec{Y}_3 \equiv \begin{bmatrix} Y_{31} \\ Y_{32} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{21} & 0 \\ g_{12} & g_{22} & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 q_1 \\ \alpha_2 q_2 \\ \alpha_3 q_3 \end{bmatrix} + \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \equiv \begin{bmatrix} \vec{g}_1 & \vec{g}_2 & \vec{0} \end{bmatrix} D\vec{q} + \vec{\nu}$$

$$\vec{Y} \equiv \begin{bmatrix} \vec{Y}_1 \\ \vec{Y}_2 \\ \vec{Y}_3 \end{bmatrix} \equiv \begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{21} \\ Y_{22} \\ Y_{31} \\ Y_{32} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & g_{21} & g_{31} \\ 0 & g_{22} & g_{32} \\ g_{11} & 0 & g_{31} \\ g_{12} & 0 & g_{32} \\ g_{11} & g_{21} & 0 \\ g_{12} & g_{22} & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 q_1 \\ \alpha_2 q_2 \\ \alpha_3 q_3 \end{bmatrix} + \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_1 \\ \nu_2 \\ \nu_1 \\ \nu_2 \end{bmatrix} \equiv$$

$$\begin{bmatrix} \vec{0} & \vec{g}_2 & \vec{g}_3 \\ \vec{g}_1 & \vec{0} & \vec{g}_3 \\ \vec{g}_1 & \vec{g}_2 & \vec{0} \end{bmatrix} D\vec{q} + \begin{bmatrix} \vec{\nu} \\ \vec{\nu} \\ \vec{\nu} \end{bmatrix}$$

III. A FIXED-POINT PROBLEM

Equation (4) can now be re-written as:

$$\frac{q_i g_{i1}}{Y_{i1}} + \cdots + \frac{q_i g_{iK}}{Y_{iK}} = 1 \qquad (11)$$

For a *fixed* interference vector $\vec{Y}$ this equation can be easily solved for q_i, to obtain the vector $\vec{q}$ which would satisfy the system of equations of the form (11). This suggests the following approach. For a given $\vec{Y}$, define the transformation:

$$\vec{T}(\vec{q}) := \begin{bmatrix} \left(\frac{g_{11}}{Y_{11}(\vec{q})} + \cdots + \frac{g_{1K}}{Y_{1K}(\vec{q})}\right)^{-1} \\ \vdots \\ \left(\frac{g_{N,1}}{Y_{N,1}(\vec{q})} + \cdots + \frac{g_{N,K}}{Y_{N,K}(\vec{q})}\right)^{-1} \end{bmatrix} \qquad (12)$$

$\vec{T}(\vec{q})$ yields the power vector under which each transceiver would achieve its desired α_i if the interference vector $\vec{Y}$ remained fixed. Of course, as the power levels are adjusted, a new interference vector results, $\vec{Y}(\vec{q}) = \mathcal{G}D\vec{q} + \vec{\boldsymbol{\nu}}$. This new vector will lead to further power adjustments, and so on, in an iterative manner.

Under the appropriate conditions, this algorithm will "converge" in the sense that a vector $\vec{q}^*$ exists such that $\vec{q}^* = T(\vec{q}^*)$; i.e., $\vec{q}^*$ is a "fixed point" of the mapping $\vec{T}$. These conditions determine the feasibility of the ratios α_i.

IV. MATHEMATICAL RESULTS

Several well-known results useful in solving fixed point problems are presented below. Some relevant background material is discussed in a mathematical appendix.

A. Brouwer's Fixed Point Theorem

Theorem(Brouwer's): Let $T : S \rightarrow S$ be a continuous function from a non-empty, compact, convex set $S \subset \Re^n$ *into* itself. There is a $x_0 \in S$ such that $x_0 = T(x_0)$.

Proof: See [1, p.28].

B. Banach's result

Contraction Mappings. Let S be a vector space endowed with the norm $\|\cdot\|$. Suppose T is a mapping from S into itself (i.e., $T : S \rightarrow S$). If there is a real number λ, $0 \leq \lambda < 1$ such that $\|T(x) - T(y)\| \leq \lambda \|x - y\|$for all x , $y \in S$ then T is said to be a *contraction mapping*.

Successive approximation. For expositional convenience, let $T^m(x)$ for $x \in S$ be defined inductively by $T^0(x) = x$ and $T^{m+1}(x) = T\left(T^m(x)\right)$, with $m \in \{1, 2, \cdots\}$.

Banach's Contraction Mapping Principle: Let S be a closed subset of $\Re^n$. Suppose that T is a mapping from S into itself. If T is a contraction mapping on S, there is a unique vector $x_0 \in S$ such that $x_0 = T(x_0)$. Moreover, x_0 can be obtained by "successive approximation", starting from an arbitrary initial $x \in S$; i.e., for all $x \in S$, $\lim_{m\to\infty} T^m(x) = x_0$.

Furthermore,

$$\|T^m(x) - x_0\| \leq \frac{\lambda^m}{1-\lambda} \|T(x) - x\|$$

Proof: See [4, Theorem 3.1, page 41].

More general versions of this result, and many extensions can be found in many sources, including [4].

Contraction condition for differentiable mappings. If the considered vector space S is *convex* and the considered mapping is such that its derivative $T'(x)$ exists over S, then for any $x_1, x_2 \in S$, and $L := \{x = x_1 + t(x_2 - x_1) : 0 \leq t \leq 1\}$ the mean value inequality holds that

$$\|T(x_1) - T(x_2)\| \leq \sup_{x \in L} \|T'(x)\| \, \|x_1 - x_2\| \qquad (13)$$

Hence, in this situation $\|T'(x)\| \leq \lambda < 1$ implies that T is a contraction mapping on S [5, p. 272].

V. FIXED POINTS, AND ALGORITHMS

A. From S into S

In order for the previously-mentioned results to be applicable to the mapping $\vec{T}(\vec{q})$, it must map vectors from an appropriate set, to vectors *in the same* set.

1) Scaled Power Set: In general, any feasible scaled power vector $\vec{q}$ must be in the set $S := \left\{\vec{q} \in \Re_+^{\mathrm{N}}, \vec{0} \leq \vec{q} \leq \vec{q}^L\right\}$ with $\vec{q}^L$ the "largest" feasible total received (scaled) power vector. If P_i^L is the transmission power limit of transceiver i, $q_i^L = (1/\alpha_i) P_i^L \sum_{k=1}^{K} h_{ik}$.

This set is closed by definition. It is straightforward to verify that it is also convex.

2) "into" condition: It is immediate that each component $T_i(\vec{q})$ is increasing in each component of $\vec{Y}_i$. And each component of $\vec{Y}_i(\vec{q})$ is increasing in $\vec{q}$. Therefore, to verify that $\vec{T}(\vec{q})$ is in S, the critical value is $\vec{T}(\vec{q}^L)$. Specifically, it is necessary that $T_i(\vec{q}^L) \leq q_i^L$ or that, (see equation (12)),

$$\frac{g_{i1} q_i^L}{Y_{i1}(\vec{q}^L)} + \cdots + \frac{g_{iK} q_i^L}{Y_{iK}(\vec{q}^L)} \geq 1 \qquad (14)$$

where, by equation (5), $Y_{ik}(\vec{q}^L) = \sum_{\substack{j=1 \\ j \neq i}}^{N} \alpha_j q_j^L g_{jk} + \nu_k$.

Recall that $P_i h_{ik} \equiv g_{ik}\alpha_i q_i$. Hence, the preceding condition can be written as:

$$\alpha_i \le \frac{P_i^L h_{i1}}{\sum_{\substack{j=1\\j\neq i}}^N P_j^L h_{j1} + \nu_1} + \cdots + \frac{P_i^L h_{iK}}{\sum_{\substack{j=1\\j\neq i}}^N P_j^L h_{jK} + \nu_K} \quad (15)$$

It may be reasonable to assume that $P_i^L = P^L \,\forall i$, and that ν_k/P^L is "very small" as compared to $\sum_{\substack{j=1\\j\neq i}}^N h_{jk}$. Then, condition (15) becomes:

$$\alpha_i \le \frac{h_{i1}}{\sum_{\substack{j=1\\j\neq i}}^N h_{j1}} + \cdots + \frac{h_{iK}}{\sum_{\substack{j=1\\j\neq i}}^N h_{jK}} \quad (16)$$

B. Existence of a fixed point

Proposition: If a vector of desired CIR, $\vec{\alpha}$, is such that condition (14) is satisfied – or so is the "neater" condition (16), under the mild assumptions under which it is valid – then $\vec{\alpha}$ is feasible.

Proof: The set S of feasible (scaled) power vectors is a closed, bounded and convex subset of $\Re^N$. If condition (14) or, when appropriate, (16) , is satisfied, the mapping $\vec{T}(\vec{q})$ is into. It is considered self-evident (and can be shown) that this mapping is continuous over the set S. Therefore, Brouwer's fixed-point theorem applies (see section IV-A). Hence, $\vec{T}(\vec{q})$ has at least one fixed point. Q.E.D.

However, Brouwer's theorem says nothing about the uniqueness of the solution, or the behavior of the algorithm discussed in section IV-B.

VI. Toward a unique fixed point

This section explores conditions under which the norm of the derivative of $\vec{T}(\vec{q})$ is less than one, so that Banach's principle can be applied. In this case, a unique fixed-point exists, and it can be found via a simple, well-behaved algorithm (see section IV-B).

A. Derivative of $\vec{T}(\vec{q})$

$\vec{T}'(\vec{q})$ is given by the corresponding "Jacobian" matrix of partial derivatives, where $\partial T_i/\partial q_j$ corresponds to its i^{th} row and j^{th} column. From equation (12),

$$T_i(\vec{q}) = \left(\frac{g_{i1}}{Y_{i1}(\vec{q})} + \cdots + \frac{g_{iK}}{Y_{iK}(\vec{q})}\right)^{-1} \quad (17)$$

Thus,

$$\frac{\partial T_i}{\partial q_j} = \frac{\partial T_i}{\partial Y_{i1}}\frac{\partial Y_{i1}}{\partial q_j} + \frac{\partial T_i}{\partial Y_{i2}}\frac{\partial Y_{i2}}{\partial q_j} + \cdots + \frac{\partial T_i}{\partial Y_{iK}}\frac{\partial Y_{iK}}{\partial q_j} \quad (18)$$

$\partial T_i/\partial Y_{ik}$ is obtained as:

$$g_{ik}Y_{ik}^{-2}\left(\frac{g_{i1}}{Y_{i1}} + \frac{g_{i2}}{Y_{i2}} + \cdots + \frac{g_{iK}}{Y_{iK}}\right)^{-2} \equiv g_{ik}\left(\frac{T_i}{Y_{ik}}\right)^2 \quad (19)$$

Additionally, by equation (5), $Y_{ik}(\vec{q}) = \sum_{\substack{j=1\\j\neq i}}^N \alpha_j q_j g_{jk} + \nu_k$. Therefore,

$$\frac{\partial Y_{ik}}{\partial q_j} = \begin{cases} 0 & \text{for } j = i \\ \alpha_j g_{jk} & \text{for } j \neq i \end{cases} \quad (20)$$

Replacing equations (19) and (20) into equation (18) one obtains that

$$\partial T_i/\partial q_i \equiv 0 \,\forall i \quad (21)$$

and, for $j \neq i$, $\partial T_i/\partial q_j =$

$$\begin{aligned} T_i^2\left(\frac{g_{i1}}{Y_{i1}^2}\frac{\partial Y_{i1}}{\partial q_j} + \frac{g_{i2}}{Y_{i2}^2}\frac{\partial Y_{i2}}{\partial q_j} + \cdots + \frac{g_{iK}}{Y_{iK}^2}\frac{\partial Y_{iK}}{\partial q_j}\right) &= \\ \alpha_j T_i^2\left(\frac{g_{i1}}{Y_{i1}^2}g_{j1} + \frac{g_{i2}}{Y_{i2}^2}g_{j2} + \cdots + \frac{g_{iK}}{Y_{iK}^2}g_{jK}\right) &= \\ \alpha_j T_i^2 \sum_{k=1}^K \frac{g_{ik}g_{jk}}{Y_{ik}^2} & \end{aligned} \quad (22)$$

B. Norm of $T'(\vec{q})$

By definition, $\left\|\vec{T}'(\vec{q})\right\|_\infty$ is the maximum absolute row sum of $\vec{T}'(\vec{q})$ (see section A). In view of equations (21) and (22), the i^{th} row of $\vec{T}'(\vec{q})$ adds up to

$$\begin{aligned} \sum_{j=1}^N \frac{\partial T_i}{\partial q_j} &= T_i^2(\vec{q})\sum_{\substack{j=1\\j\neq i}}^N \alpha_j \sum_{k=1}^K \frac{g_{ik}g_{jk}}{Y_{ik}^2(\vec{q})} \\ &= T_i^2(\vec{q})\sum_{k=1}^K \frac{g_{ik}}{Y_{ik}^2(\vec{q})}\sum_{\substack{j=1\\j\neq i}}^N \alpha_j g_{jk} \\ &:= f_{ik}(\vec{q})\rho_{ik} \end{aligned} \quad (23)$$

Observe that $\rho_{ik} := \sum_{j=1}^N \alpha_j g_{jk} - \alpha_i g_{ik}$ is the sum of the components of $\hat{\mathcal{G}}^{ik}$, which is the row of the matrix $\mathcal{G}D \equiv \hat{\mathcal{G}}$ associated with Y_{ik} (see equation (10)). It represents the parameters in equation (23) which can be influenced by limiting the vector $\vec{\alpha}$. For a given $\vec{q}$, the function $f_{ik}(\vec{q}) := T_i^2(\vec{q})\sum_{k=1}^K g_{ik}/Y_{ik}^2(\vec{q})$ is determined by the channel via the various path loss coefficients.

C. Contraction condition

On the basis of the preceding development, in order for $\left\|\vec{T}'(\vec{q})\right\|_\infty < 1$ so that $\vec{T}(\vec{q})$ is a contraction, $\vec{\alpha}$ must be such that

$$\max_{\vec{q}} \; f_{ik}(\vec{q})\rho_{ik} < 1 \;\; \forall i, k \quad (24)$$

with ρ_{ik} the sum of the components of $\hat{\mathcal{G}}^{ik}$(see equation (10)) and $f_{ik}(\vec{q})$ given by:

$$f_{ik}(\vec{q}) = \frac{\frac{g_{i1}}{Y_{i1}^2} + \cdots + \frac{g_{iK}}{Y_{iK}^2}}{\left(\frac{g_{i1}}{Y_{i1}} + \cdots + \frac{g_{iK}}{Y_{iK}}\right)^2} \quad (25)$$

D. Properties of the Contraction Condition

1) **Well-definedness**. Condition (24) is well defined, because f_{ik} is a continuous function, for which it must have a maximum over a closed and bounded set (see sec. (V-A.1))
2) $f_{ik} \ge 1$. This is so because f_{ik} is of the form $(\lambda_1\phi(x_1) + \cdots + \lambda_K\phi(x_K))/\phi(\lambda_1 x_1 + \cdots + \lambda_K x_K)$ with $\phi(x) = x^2$, $\lambda_i \in [0, 1]$, $\sum_i \lambda_i = 1$ and x_i positive.

The function $\phi(x) = x^2$ is easily shown to by convex. And for any convex ϕ, Jensen's inequality holds that $\lambda_1\phi(x_1) + \cdots + \lambda_K\phi(x_K) \geq \phi(\lambda_1 x_1 + \cdots + \lambda_K x_K)$. (See also section B in the appendix).

3) If $\vec{q}$ is such that $Y_{ik}(\vec{q}) = Y_{il}(\vec{q})\ \forall k, l$ then $f_{ik}(\vec{q}) = 1$. This follows directly because $\sum_k g_{ik} = 1$ by definition (see equation (3))
4) If each transceiver is "equidistant" to each "receiver" (antenna), in the sense that $h_{ik} = h_{il}\ \forall i, k, l$ then $f_{ik}(\vec{q}) \equiv 1$. This also follows directly because in this case $g_{ik} = g_{il} \equiv 1/K\ \forall i, k, l$ (see equation (3)). In this case, the contraction condition (24) reduces to $\|\mathcal{G}D\|_\infty \equiv \|\mathcal{G}\vec{\alpha}\|_\infty < 1$
5) In the special case in which K=2, the maximum f_{ik} is attained for the particular $\vec{q}$ which creates the largest "separation" between Y_{i1} and Y_{i2}. (See section B in the appendix).

E. A unique solution and an algorithm to find it

Proposition: If a vector of desired CIR, $\vec{\alpha}$, is such that condition (14) , or, when appropriate, condition (16), is satisfied, and so is condition (24) above, then $\vec{\alpha}$ is feasible. Furthermore, the power vector leading to $\vec{\alpha}$ is unique, and can be obtained via the well-behaved algorithm described in section IV-B.

Proof: The "power set" S is a closed subset of $\Re^n$ (see section V-A.1). If condition (14) , or, when appropriate, condition (16), is satisfied, the transformation $T(\vec{q})$ is a mapping from S into S. If condition (24) is also satisfied, T is a contraction mapping. Therefore, under the hypothesis of this proposition, Banach's principle applies (see section IV-B). Q.E.D.

VII. Discussion

This note provides an answer to the question of whether a certain vector, $\vec{\alpha}$, of positive numbers interpreted as desired "carrier-to-interference ratios" is feasible in a macrodiversity CDMA environment, in the sense that there are feasible power levels which produce the desired ratios. The answer is in the affirmative whenever condition (14) is satisfied. Under mild assumptions, this condition takes the simple form $\alpha_i \leq A_i$, with A_i a relatively simple function involving ratios of the various path loss coefficients of the active transceivers. However, not much can be said about the underlying power vector, or the performance of any particular algorithm in finding it.

This note also explores a more elaborate condition, (24). *Together* with condition (14), condition (24) implies that the power vector leading to $\vec{\alpha}$ is unique, and can be found by way of a well-behaved simple algorithm. This algorithm can depart from an arbitrary power vector. It is of the form $x^{n+1} = f(x^n)$ with x^0 arbitrary. A simple expression gives the "error" after a given number of iterations.

In general, condition (24) depends on the maximum of a relatively simple function. More research is needed to determine the practical implications of obtaining this maximum, or a reasonable approximation for it. However, in special cases, in particular when each terminal happens to be "equidistant" from the antennas, this condition reduces to $\|\mathcal{G}\vec{\alpha}\|_\infty < 1$. In words, this condition requires that the "largest weighted average" of the desired α_i's be less than one. The possible weight vectors are the rows of the "relative gains" matrix $\mathcal{G}$. It is significant that each row of this matrix always has at least one element equal to zero, which implies that, in verifying this condition, at most N-1 of the α_i's are simultaneously weighted.

Space limitations preclude a comprehensive contrasting of these results to those originally presented in [2]. Nevertheless, some brief comments will be made.

First, condition (14) does not have an obvious "counterpart" in [2]. The result derived in [2] under the "self-interference" approximation is $\sum_{i=1}^N \alpha_i < K$, which limits the sum without imposing an individual limit on each term. However, one can make a rough comparison by assuming that condition (16) applies and is satisfied by each i, and that each terminal is "equidistant" from each antenna so that $h_{ik} \approx h_{il} \approx h_i\ \forall i, k, l$. This symmetry would practically arise, for example, with $K = 2$, if the two receiving antennas are directly across from each other in opposing sides of a road segment, and each terminal is located along the axis of this segment. When this symmetry exists,

$$\sum_{i=1}^N \alpha_i \leq \sum_{i=1}^N K \frac{h_i}{\sum_{\substack{j=1 \\ j\neq i}}^N h_j} \approx K \frac{\sum_{i=1}^N h_i}{\sum_{\substack{j=1 \\ j\neq i}}^N h_j} > K$$

This indicates that condition (16) is "less conservative" that the approximated condition from [2].

The more elaborate condition, (24), may also be compared, with caution, with the approximated result from [2], by considering, again, the special symmetric situation. In this case, condition (24) reduces to $\|\mathcal{G}\vec{\alpha}\|_\infty < 1$, as remarked above. Additionally, each $g_{ik} = 1/K$ (see equation (3)). Therefore, the j^{th} row of this matrix has the form $(1/K)\begin{bmatrix} 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \end{bmatrix}$ where the only zero is at the j^{th} position (see equation (8)). Hence, the product of the j^{th} row of $\mathcal{G}$ by $\vec{\alpha}$ simply adds all the components of $\vec{\alpha}$ except for α_j and divide the sum by K. For example, with 3 terminals, the second row of $\mathcal{G}$ is $(1/K)\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$ and the product of this row by $\vec{\alpha}$ equals $(\alpha_1 + \alpha_3)/K$. $\|\mathcal{G}\vec{\alpha}\|_\infty$ simply picks out the largest component of the product $\mathcal{G}\vec{\alpha}$. The j^{th} component of $\mathcal{G}\vec{\alpha}$ is a sum of the form $(\sum_{i=1}^N \alpha_i - \alpha_j)/K$. Thus, the largest component of $\mathcal{G}\vec{\alpha}$ will be the one that leaves out of the sum the smallest component of $\vec{\alpha}$. For instance, if α_N happens to be the smallest α_i, then $\|\mathcal{G}\vec{\alpha}\|_\infty = (1/K)\sum_{i=1}^{N-1} \alpha_i$. Hence, in the "symmetric" case, the approximated result demands that $\sum_{i=1}^N \alpha_i < K$, whereas condition (24) only imposes that $\sum_{i=1}^{N-1} \alpha_i < K$ (assuming α_N is the smallest desired CIR).

It is stressed that, in the context of a 3G network, when relatively few high data-rate terminals may be sharing a channel, the less conservative results could make a significant difference. For example, suppose K=1, and that three high data-rate sources wish to share a channel, each demanding a CIR of 2/5. This is plausible in a VSG-CDMA situation (see introduction). The approximated result dictates that only 2 of them can be accommodated, whereas condition (24) indicates that all three can, "with room to spare". In a 3G environment, leaving, unnecessarily, out even one terminal could be significant, if, as presumed, the additional terminal would have transmitted megabits of data each second.

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APPENDIX

A. Background material

Let S denote a vector space (for a formal definition of these spaces see [5, pp. 11-12]).

Norms and metrics. A norm, $\|\cdot\|$, on S is a function from S into the non-negative real numbers $\Re_+$ "generalizing" the idea of the "Euclidean length" of a vector. It engenders a "metric" ('distance'), defined as $d(x, y) = \|x - y\|$.

Infinity norm. $\|\cdot\|_\infty$ is defined as

$$\|\vec{x}\|_\infty := \max(|x_1|, |x_2|, \cdots, |x_N|) \tag{26}$$

Linear operators. If T is a mapping from a vector space, S_1, into another, S_2, (i.e., $T : S_1 \to S_2$), it is said to be *linear* if for any x , $y \in S_1$ and λ_1 , $\lambda_2 \in \Re$, $T(\lambda_1 x + \lambda_2 y) = \lambda_1 T(x) + \lambda_2 T(y)$.

The **operator norm** of a linear operator T is defined as

$$\|T\| := \sup_{\|x\| \neq 0} \frac{\|T(x)\|}{\|x\|} \equiv \sup_{\|x\|=1} \|T(x)\| \tag{27}$$

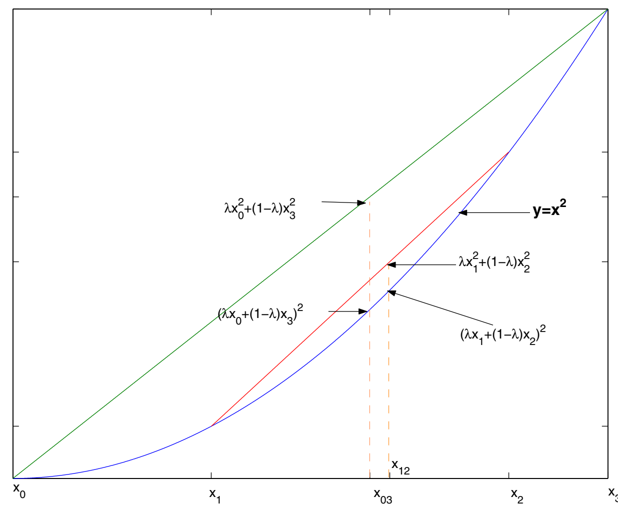
where sup denotes the supremum or least upper bound.

Matrix infinity norm. When a linear operator is expressed as $T(x) = Ax$, with A a suitably dimensioned matrix, and the underlying norm is $\|\cdot\|_\infty$, its "operator norm" is the "maximum absolute row sum" of A. If a_{ij} denotes the element corresponding to the i^{th} row and j^{th} column of matrix A,

$$\|A\|_\infty := \sup_{\|x\|=1} |Ax| = \max_i \left(\sum_j |a_{ij}| \right) \tag{28}$$

Row sum of the product of two non-negative matrices. If A and B are suitably dimensioned non-negative matrices, the row sum of the product, AB, can be obtained as the product $A \cdot \text{rsum}(B)$, with $\text{rsum}(B)$ the vector resulting from the sum of the columns of B.

B. Maximum of an interesting ratio



$\lambda x_i^2 + (1-\lambda)x_j^2$ versus $(\lambda x_i + (1-\lambda)x_j)^2$

It is of interest to determine a supremum of the form

$$\sup_{0 \le x_0 \le x_1, x_2 \le x_3} \frac{\lambda x_1^2 + (1-\lambda)x_2^2}{(\lambda x_1 + (1-\lambda)x_2)^2} \tag{29}$$

where $0 \le \lambda \le 1$ is fixed, and $x_1 \le x_2$ are positive real numbers in certain interval.

The above ratio is a continuous function for which it must necessarily have a maximum over any closed and bounded set.

Also, $x_{12} := \lambda x_1 + (1-\lambda)x_2$ is simply a convex combination ("mixture") of x_1 and x_2; i.e., a point between x_1 and x_2. Likewise, $\lambda x_1^2 + (1-\lambda)x_2^2$ is a "mixture" of x_1^2 and x_2^2, with the same "mixture" parameter λ (see figure (B)).

The function $f(x) := x^2$ is easily shown to be convex. And, by definition, any convex function satisfies $\lambda f(x_1) + (1-\lambda)f(x_2) \ge f(\lambda x_1 + (1-\lambda)x_2)$. Therefore, the ratio (29) is always greater than or equal to 1.

It is straightforward to verify that the first-order optimizing conditions for this ratio are satisfied whenever $x_1 = x_2$. But in this case, the ratio equals 1, which is its smallest possible value. Therefore, the maximum is attained over the boundary of the feasible region; i.e., $x_1 = x_0$ and $x_2 = x_3$ leads to the maximum.