Generalised multi-receiver radio network: Capacity and asymptotic stability of power control through Banach's fixed-point theorem

Virgilio RODRIGUEZ, R. Mathar, A. Schmeink

Theoretische Informationstechnik
RWTH Aachen
Aachen, Germany
email: vr@ieee.org

IEEE WCNC, Budapest, 8 April 2009

Outline

1. Power, interference and QoS: Issues
2. General models of radio network
3. Technical development and results
4. Comparative case study: Macro-diversity
5. Conclusions

Power, interference and QoS: 2 questions

- A user's quality of service (QoS) increases with the power in its signal at its receiver(S), and decreases with the interfering power present at the concerned receiver(S)
- Typically each terminal "aims" for certain level of QoS
- For fixed interference, if the terminal power limit is "very high" then it can set the "right" power level
- But interference to a terminal grows with the power emitted by the others.
- Even without power limits, it is unclear that each terminal can achieve its desired QoS.
- Two fundamental questions:
  - Are the QoS targets feasible (achievable)? $\Leftarrow$CRITICAL for admission control!
  - If yes, which power vector achieves the QoS targets?

Abstract model (Yates'95)

- N terminals whose power choices affect each other
- Terminal i chooses a power p_i given by a function $g_i(\mathbf{p}_{-i})$, with $\mathbf{p}_{-i}$ denoting the power levels of the others
- $p_i = g_i(\mathbf{p}_{-i})$ leads to terminal i its desired QoS for given $\mathbf{p}_{-i}$
- All details of the network (the QoS targets, number of receivers, interference functions, etc) are assumed "hidden" inside the power functions
- These functions are assumed to satisfy some simple mathematical properties (monotonicity, homogeneity, etc)
- Considering the functions properties the analyst addresses some of the fundamental questions about QoS achievability[1]

Generalised multi-receiver radio network

- N transmitters, K receivers
- i's QoS requirement given by

$$\mathcal{Q}_i\left(\frac{P_i h_{i,1}}{\mathcal{Y}_{i,1}(\mathbf{P})+\sigma_1},\cdots,\frac{P_i h_{i,K}}{\mathcal{Y}_{i,K}(\mathbf{P})+\sigma_K}\right)\geq\kappa_i \qquad (1)$$

- $h_{i,k}$ is the known channel gain from TX i to RX k
- $\mathcal{Q}_i$, and $\mathcal{Y}_{i,k}$ are general functions obeying certain simple properties (monotonicity, homogeneity, etc)

An example: macro-diversity

- macro-diversity:
  - definition
    - cellular structure is removed
    - all transmitters are jointly decoded by all receivers
    - equivalently, 'one cell' with a distributed antenna array
  - *i*'s QoS is given by [2]:
    - $P_i h_{i,1}/(Y_{i,1}+\sigma_1)+\cdots+P_i h_{i,K}/(Y_{i,K}+\sigma_K)$
    - with $Y_{i,k}=\sum_{n\neq i} P_n h_{n,k}$
  - Thus, $\mathscr{Y}_{i,k}(\mathbf{P})=\sum_{n\neq i} P_n h_{n,k}$ and
  - $\mathscr{Q}_i(\mathbf{x})=\mathscr{Q}^{\mathrm{MD}}(\mathbf{x})=x_1+\cdots+x_K$
    (notice that same function works for all *i*)
- Other examples: all scenarios from (Yates 1995)[1]

Why a new model?

- Both models can be useful (think macroeconomics vs. microeconomics)
- Abstract model is more general (powerful?)
- Detailed model
  - is closer to 'real' world (easier to interpret)
  - separates QoS function from Interference function (conceptually different... may have different properties)
  - may provide insights/opportunities not otherwise available (e.g., we provide a simple closed-form solution for this model!... see ICC'09)

Key technical step

- If $\mathscr{Q}_i$ is non-decreasing, replace QoS constraint with:

$$\mathscr{Q}_i\left(\frac{P_i h_{i,1}}{\|\mathbf{Y_i}\|_\infty + \|\sigma\|_\infty}, \cdots, \frac{P_i h_{i,K}}{\|\mathbf{Y_i}\|_\infty + \|\sigma\|_\infty}\right) \geq \kappa_i \qquad (2)$$

- where

$$\|\mathbf{Y}_i\|_\infty \equiv \max_k \{ Y_{i,k} \} \qquad (3)$$

$$\|\sigma\|_\infty \equiv \max_k \{\sigma_k\} \qquad (4)$$

- If $\mathscr{Q}_i$ is homogeneous (i.e., $\mathscr{Q}_i(\lambda \mathbf{x}) = \lambda \mathscr{Q}_i(\mathbf{x}) \; \forall \lambda \in \Re_+$), (2) becomes:

$$\frac{P_i \mathscr{Q}_i\left(h_{i,1}, \cdots, h_{i,K}\right)}{\|\mathbf{Y_i}\|_\infty + \|\sigma\|_\infty} := \frac{P_i h_i}{\|\mathbf{Y_i}\|_\infty + \|\sigma\|_\infty} \geq \kappa_i \qquad (5)$$

$$h_i := \mathscr{Q}_i\left(h_{i,1}, \cdots, h_{i,K}\right) \qquad (6)$$

Power adjustment process

the power adjustment process can be written as:

$$P_i^{t+1} = \frac{\kappa_i}{h_i}\left(\left\|\left(\mathscr{Y}_{i,1}(\mathbf{P}^t), \cdots, \mathscr{Y}_{i,K}(\mathbf{P}^t)\right)\right\|_\infty + \|\sigma\|_\infty\right) \tag{7}$$

Or, with the change of variable:

$$p_i := \frac{h_i P_i}{\kappa_i} \tag{8}$$

Then, the adjustment becomes

$$p_i^{t+1} = f_i(\mathbf{p}^t) + \|\sigma\|_\infty \tag{9}$$

where,

$$f_i(\mathbf{p}) := \|\mathbf{Y_i}\|_\infty \equiv \left\|\left(\mathscr{Y}_{i,1}(\mathbf{p}), \cdots, \mathscr{Y}_{i,K}(\mathbf{p})\right)\right\|_\infty \tag{10}$$

Methodology: Fixed-point theory

- power adjustment process $\Rightarrow$a *transformation* $\mathbf{T}$ that takes a vector $\mathbf{x}$ and "converts" it into a new one, $\mathbf{T}(\mathbf{x})$.
- A limit is a $\mathbf{x}^*$ s.t. $\mathbf{x}^* = \mathbf{T}(\mathbf{x}^*)$ (a "fixed-point" of $\mathbf{T}$)

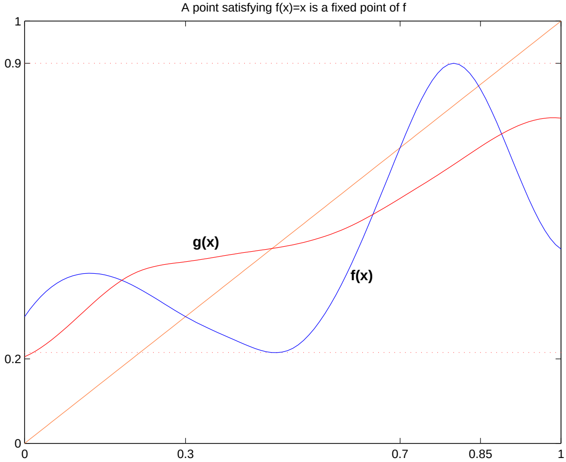
Fact

(Banach's)If $\mathbf{T} : S \to S$ *is a contraction in* $S \subset \Re^M$ *(that is,* $\exists r \in [0,1)$ *such that* $\forall \mathbf{x}, \mathbf{y} \in S, \| T(\mathbf{x}) - T(\mathbf{y}) \| \leq r \| \mathbf{x} - \mathbf{y} \|$ *) then* $\mathbf{T}$ *has a unique fixed-point, that can be found by successive approximation, irrespective of the starting point .[3]*

Fact

With $\mathbf{x}_1 := T(\mathbf{x}_0), \ldots, \mathbf{x}_m := T(\mathbf{x}_{m-1})$,
then $\| \mathbf{x}_n - \mathbf{x}^* \| \leq (r^n/(1-r)) \| \mathbf{x}_1 - \mathbf{x}_0 \|$ *[3]*

Fixed points in $\Re$



Key technical result

- If $\mathscr{Y}_{i,k}(\cdot)$ is a (semi)norm, then $\left\| \left(\mathscr{Y}_{i,1}(\mathbf{p}), \cdots, \mathscr{Y}_{i,K}(\mathbf{p})\right) \right\|_\infty$ is also a (semi)norm, and satisfies the "triangle inequality".
- This helps us prove our main result:

Theorem

If each $\mathscr{Y}_{i,k}$ (i) is a semi-norm, (ii) and $\mathscr{Y}_{i,k}(\mathbf{x}) \leq \mathscr{Y}_{i,k}\left(\|\mathbf{x}\|_\infty \mathbf{1}_N\right) \quad \forall \mathbf{x} \in \Re^N$ (weak monotonicity), and (iii) $\mathscr{Y}_{i,k}(\mathbf{1}_N) < 1$, then the transformation $\mathbf{T}$ defined by $T_i(\mathbf{p}) = \left\| \left(\mathscr{Y}_{i,1}(\mathbf{p}), \cdots, \mathscr{Y}_{i,K}(\mathbf{p})\right) \right\|_\infty + \|\sigma\|_\infty$ for $\mathbf{p} \in \Re^N$, $i = 1 \ldots N$ is a contraction.

Applications

- The key feasibility condition: $\mathscr{Y}_{i,k}(\mathbf{p}^*) < 1$ with $p_i^* = h_i P_i / \kappa_i = 1$ (i.e., $P_i^* = \kappa_i / h_i$), then each κ_i is feasible.
- For $\mathscr{Y}_{i,k}(\mathbf{P}) = \sum_{n \neq i} P_n h_{n,k}$, $\mathscr{Y}_{i,k}(\mathbf{p}) := \sum_{\substack{n=1 \\ n \neq i}}^{N} (h_{i,k}/h_i) \kappa_n p_n$ and

  Theorem 3 leads to

$$\sum_{\substack{n=1 \\ n \neq i}}^{N} \frac{h_{i,k}}{\mathscr{Q}_i \left(h_{i,1}, \cdots, h_{i,K} \right)} \kappa_n < 1 \quad \forall i, k \tag{11}$$

- Interpretation: greatest weighted sum of $N-1$ carrier-to-interference ratios < 1
- For macro-diversity, $\mathscr{Q}_i \left(h_{i,1}, \cdots, h_{i,K} \right) = \sum_k h_{n,k}$

Original feasibility condition

- (Hanly, 1996 [2]) provides the condition

$$\sum_{n=1}^{N} \kappa_n < K$$

- Formula derived under certain simplifying assumptions:
  - A TX contributes to own interference
  - all TX's can be "heard" by all RX's
  - non-overcrowding
- Under certain practical situations condition is counter-intuitive:
  - If there are 2 TX near *each* RX, it must be "better", than if all TX's congregate near same receiver
  - In latter case, system should behave like a one-RX system
  - But formula is insensitive to channel gains: cannot adapt!

Special symmetric scenario

- Our condition is most similar to original when $h_{i,k} \approx h_{i,m}$ for all i, k, m, in which case $g_{i,k} \approx 1/K$
- Example: TX along a road; the axis of the 2 symmetrically placed RX is perpendicular to road
- Under this symmetry (and with $\kappa_N \leq \kappa_n \forall n$ for convenience) our condition simplifies to

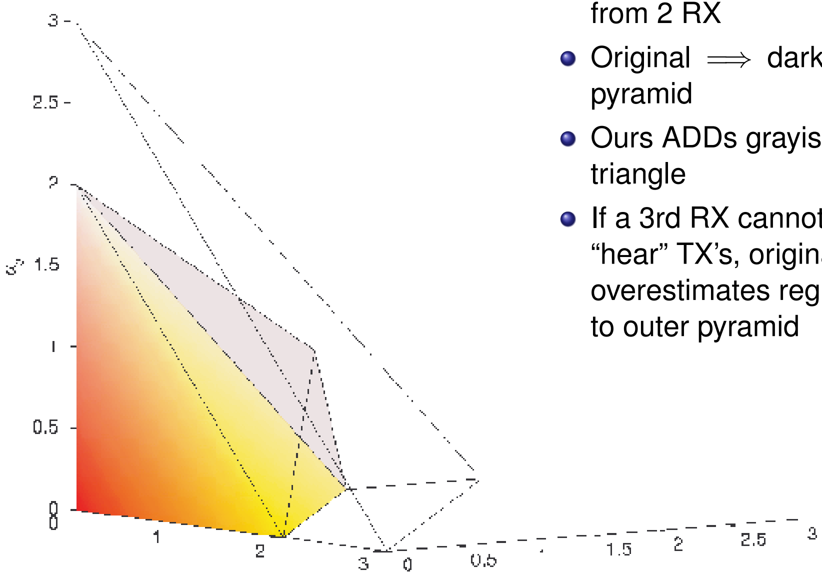
$$\sum_{n=1}^{N-1} \kappa_n < K$$

- Smallest κ is left out of sum $\implies$ our condition is less conservative than original

Partial symmetry: one receiver “too far”

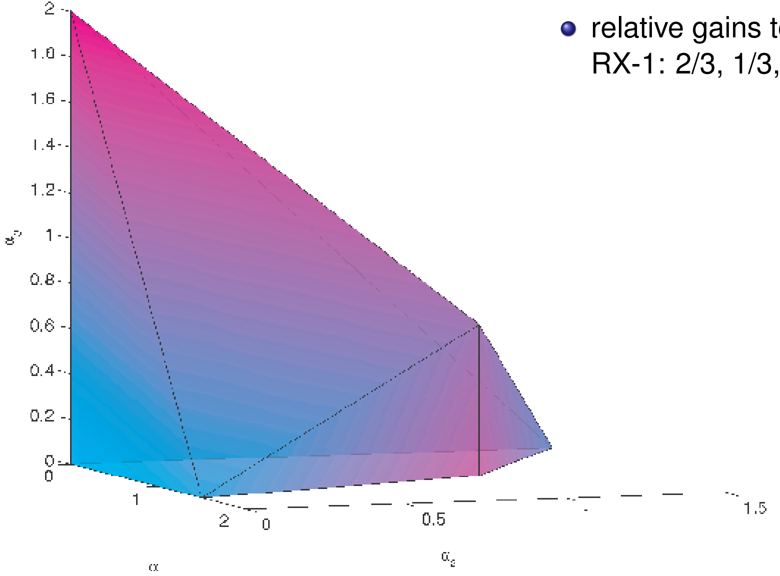
- If $K = 3$ and $h_{i,k} \approx h_{i,m}$ for all i, k, m, $g_{i,k} \approx 1/3$ and our condition becomes $\sum_{n=1}^{N-1} \kappa_n < 3$
- But suppose that $h_{i,1} \approx h_{i,2}$ but $h_{i,3} \approx 0$ (3rd receiver is “too far”), then $g_{i,3} \approx 0$ and $g_{i,1} \approx g_{i,2} \approx 1/2$
- Thus our condition leads to $\sum_{n=1}^{N-1} \kappa_n < 2$
- Our condition automatically “adapts”, whereas original remains at $\sum_{n=1}^{N} \kappa_n < 3$
- Original can over-estimate capacity if applied when some RX’s are “out of range” (because under this situation — of practical interest — some assumptions underlying the original are not satisfied)

Symmetric 3TX, 2RX scenario



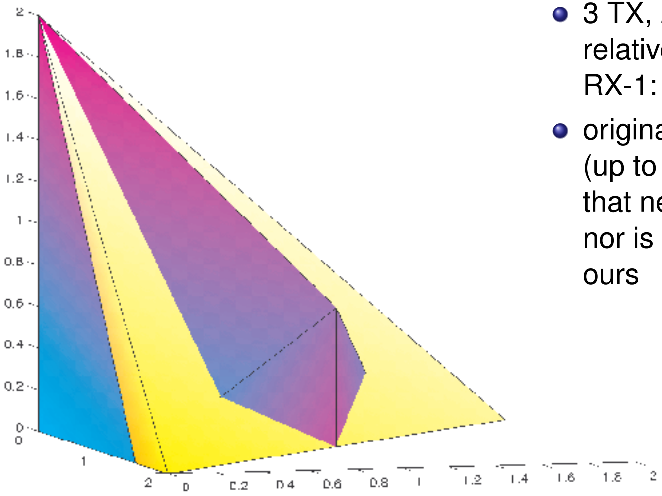
- 3 TX "equidistant" from 2 RX
- Original $\Longrightarrow$ darker pyramid
- Ours ADDs grayish triangle
- If a 3rd RX cannot "hear" TX's, original overestimates region to outer pyramid

Asymmetric 3TX, 2RX scenario: our region



- 3 TX, 2 RX
- relative gains to RX-1: 2/3, 1/3, 1/2

Asymmetric 3TX, 2RX: ours vs original



- 3 TX, 2 RX with relative gains to RX-1: 2/3, 1/3, 1/2
- original yields region (up to yellow volume) that neither contains nor is contained by ours

Summary

- Model seems to be new
- Explicit (conservative) feasibility condition given: $\mathscr{Y}_{i,k}(\mathbf{P}^*) < 1$ with $P_i^* = \kappa_i / h_i \equiv \kappa_i / \mathscr{Q}_i(h_{i,1}, \ldots, h_{i,K})$
- Solution is technology/application independent (useful for present and future networks) provided that the key functions have the assumed properties:
  - $\mathscr{Q}_i$ is non-decreasing and homogeneous
  - $\mathscr{Y}_{i,k}$ is a non-decreasing semi-norm
- The application to the special case of macro-diversity yields a result that has significant advantages over the one previously available.

Outlook

- A closed-form solution is given, as long as the key functions are non-decreasing and homogeneous (not sub-additivity assumed):
  - Let q_i be i's QoS when $\mathbf{P} = \mathbf{1}$; then if $\kappa_i \leq q_i$, $P_i^* = \kappa_i / q_i$ is a (conservative) solution.
  - Network can be *conservatively* replaced by a set of N independent TX-RX pairs.
  - See ICC'09
- Above neglects noise. However, noise has been added in another paper (Globecom'09?)
- Remember Yates' model? Similar analysis yields a feasibility-condition similar to Yates', but derived under different assumptions ==> certain functions excluded by Yates' could meet our requirements!!

Outlook II

- Homogeneity plays a key role: Can it be removed, so that only some form of monotonicity remains?
- Can/should media-based communication (e.g. video) be explicitly considered (e.g. through $\mathscr{Q}_i$)?
- Should channel gains be random?

Questions?

Main result

Let κ_i denote i's QoS target, and
$p_i = \mathcal{Q}_i\left(h_{i,1}/\mathcal{Y}_{i,1}(\mathbf{1}), \cdots, h_{i,K}/\mathcal{Y}_{i,K}(\mathbf{1})\right)$.

Theorem

If the functions $\mathcal{Q}_i$ and $\mathcal{Y}_{i,k}$ are non-negative and quasi-non-decreasing, and additionally each $\mathcal{Y}_{i,k}$ is quasi-sub-homogeneous, and each $\mathcal{Q}_i$ is super-homogeneous, and random noise is negligible, then $\kappa_i \leq p_i \; \forall i$ implies that (i) each QoS target can be achieved, in particular, (ii) with the power levels $P_i^* = \kappa_i / p_i$.

A key technical result

Let $f : \Re^M \rightarrow \Re$, and $\mathbf{1}_M$ denote the “all ones” M-vector.

Definition

f if *positively quasi-sub-homogeneous* (of degree one) if for all $r \in \Re_+$, $f(r\mathbf{1}) \leq rf(\mathbf{1})$
(if “super” replaces “sub” then “$\geq$” replaces “$\leq$”)

Definition

f if *quasi-non-decreasing if* $f(\mathbf{x}) \leq f(\|\mathbf{x}\|\mathbf{1})$, where $\|\mathbf{x}\|$ denotes the largest absolute value of the components of $\mathbf{x}$.

Fact

If f satisfies both definitions, $f(\mathbf{x}) \leq f(\|\mathbf{x}\|\mathbf{1}) \leq \|\mathbf{x}\| f(\mathbf{1})$

Argument I

Let $\mathbf{P}$ denote power (and suppose $K = 2$). $(P_i / \|\mathbf{P}\|)p_i \geq \kappa_i \equiv$

- $$(P_i / \|\mathbf{P}\|)\mathscr{Q}_i \left(\frac{h_{i,1}}{\mathscr{Y}_{i,1}(\mathbf{1})}, \frac{h_{i,2}}{\mathscr{Y}_{i,2}(\mathbf{1})} \right) \geq \kappa_i \quad \Longrightarrow$$
  $$\mathscr{Q}_i \left(\frac{P_i h_{i,1}}{\|\mathbf{P}\| \mathscr{Y}_{i,1}(\mathbf{1})}, \frac{P_i h_{i,2}}{\|\mathbf{P}\| \mathscr{Y}_{i,2}(\mathbf{1})} \right) \geq \kappa_i \qquad \text{(by homogeneity)}$$
- $\mathscr{Y}_{i,k}(\mathbf{P}) \leq \mathscr{Y}_{i,k}(\|\mathbf{P}\| \mathbf{1}) \leq \|\mathbf{P}\| \mathscr{Y}_{i,k}(\mathbf{1})$ (key Fact), thus
- $$\mathscr{Q}_i \left(\frac{P_i h_{i,1}}{\mathscr{Y}_{i,1}(\mathbf{P})}, \frac{P_i h_{i,2}}{\mathscr{Y}_{i,2}(\mathbf{P})} \right) \geq \mathscr{Q}_i \left(\frac{P_i h_{i,1}}{\|\mathbf{P}\| \mathscr{Y}_{i,1}(\mathbf{1})}, \frac{P_i h_{i,2}}{\|\mathbf{P}\| \mathscr{Y}_{i,2}(\mathbf{1})} \right)$$
- $\therefore$ if $(P_i / \|\mathbf{P}\|)p_i \geq \kappa_i$ or $P_i / \|\mathbf{P}\| \geq \kappa_i / p_i$, each κ_i is reached or exceeded

Argument II

- But $P_i \leq \|\mathbf{P}\| \ \forall i$, for any $\mathbf{P}$, by definition.
- Therefore, *no* power vector can satisfy $P_j / \|\mathbf{P}\| \geq \kappa_j / p_j > 1$
- With $\hat{\kappa} := (\kappa_1/p_1, \cdots, \kappa_N/p_N) := (\hat{\kappa}_1, \cdots, \hat{\kappa}_K)$,
  $\hat{\kappa}_i = \kappa_i / p_i \leq 1 \ \forall i \implies \|\hat{\kappa}\| \leq 1 \implies \hat{\kappa}_i / \|\hat{\kappa}\| \geq \hat{\kappa}_i \ \forall i$
- $\therefore \mathbf{P}^* = \hat{\kappa}$ satisfies $P_i / \|\mathbf{P}\| \geq \kappa_i / p_i \ \forall i$ and yields or exceeds the desired QoS

Network simplification

Consider ‘network’ with N independent (orthogonal) transmitter-receiver pairs.
Each transmitter has a power limit $\bar{P}_i = \sigma_i := 1$ and wants QoS (SNR) of κ_i.
Let the channel gain of transmitter i be $h_i := p_i$.

- The maximal QoS that i can achieve is $\bar{P}_i h_i / \sigma_i = h_i = p_i$.
- Thus κ_i is achievable provided $\kappa_i \leq p_i$.
- Furthermore, if $\kappa_i / p_i \leq 1$ then $P_i = \kappa_i / p_i$ is feasible ($\leq \bar{P}_i = 1$), and yields an SNR exactly equal to κ_i.
- The “solution” to this simple ‘network´ works for the original one!

Abstract model: Seminal work

- (Yates, 1995) [1] is the seminal work.
- It does *not* provide conditions for the feasibility of QoS targets
- It addresses the power level question indirectly by focusing on “greedy” power adjustment (terminals take turns each choosing a power level for present level of interference)
- Specifically (Yates, 1995) shows that,
  - if the the inherent QoS targets are feasible
  - and the power adjustment functions are monotonic and homogeneous, then
  - “greedy” power adjustment always converges to a unique vector, regardless of the initial power levels

Our main result

Definition

A function f is *quasi-semi-normal* if it satisfies: (i) non-negativity, (ii) quasi-non-decreasingness, (iii) quasi-sub-homogeneity and (iv) the "triangle inequality" (i.e., sub-additivity: $f(\mathbf{x}+\mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$).

Our main Theorem can be *informally* re-stated as:

Theorem

If f_i is quasi-semi-normal and satisfies $f_i(\mathbf{1}) < 1$ then the adjustment process defined by $p_i(t+1) = f_i(\mathbf{p}_{-i}(t)) + c_i$ ($c_i \geq 0$) converges to a unique power vector $\mathbf{p}^*$, regardless of the initial power levels.

Applicability of our result

- From the set of conditions $f_i(\vec{1}) < 1$ one can determine the feasibility of QoS targets
- A very large family of functions are q-semi-normal including
  - *all* known (semi-)norms
  - new (semi-)norms obtained by performing certain simple operations on known ones (e.g., the sum or maximum of two norms is a new norm)
- Three general use-cases
  - the system's "natural" power adjustment functions are q-semi-normal (e.g., the fixed assignment scenario of [1])
  - the engineer can freely impose the f_i's
  - the system can be analysed under a q-semi-normal f_i that slightly overestimates "true" power needs

Other power-control frameworks

Framework	Monotonicity	Homogeneity
Yates[1]	$\mathbf{x} \geq \mathbf{y} \implies \mathbf{f}(\mathbf{x}) \geq \mathbf{f}(\mathbf{y})$	$\lambda > 1 \implies \mathbf{f}(\lambda \mathbf{x}) < \lambda \mathbf{f}(\mathbf{x})$
S-B[4]	$\mathbf{x} \geq \mathbf{y} \implies f(\mathbf{x}) \geq f(\mathbf{y})$	$\lambda \geq 0 \implies f(\lambda \mathbf{x}) = \lambda f(\mathbf{x})$
Ours	$f(\mathbf{x}) \leq f(\lVert \mathbf{x} \rVert_\infty \mathbf{1})$	$\lambda \in (0,1) \implies f(\lambda \mathbf{1}) \leq \lambda f(\mathbf{1})$

- We add the triangle inequality, $f(\mathbf{x}+\mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$, but provide the feasibility condition $f_i(\mathbf{1}) < 1$.
- Recall that if g_i is homogeneous and monotonic then $g_i(\mathbf{x}) \leq g_i(\mathbf{1}) \lVert \mathbf{x} \rVert := \phi_i(\mathbf{x})$
- $\phi_i(\mathbf{x})$ is a multiple of a norm, and hence q-semi-normal.
- Thus one can replace g_i with ϕ_i and obtain the conservative feasibility condition $\phi_i(\mathbf{1}) \equiv g_i(\mathbf{1}) < 1$

Methodology: Fixed-point theory

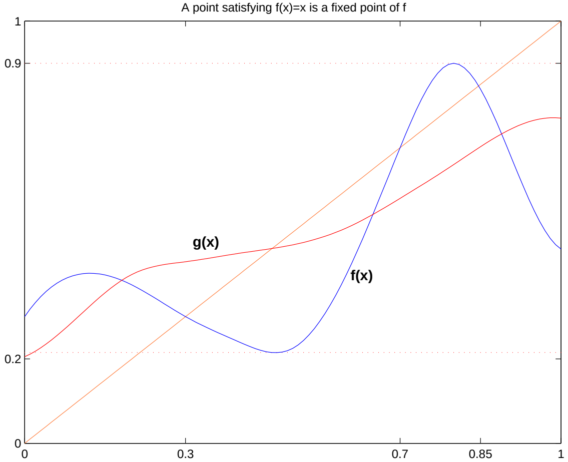
- power adjustment process $\Rightarrow$a *transformation* **T** that takes a power vector **p** and “converts” it into a new one, $\mathbf{T}(\mathbf{p})$.
- A limit of the process is a vector s.t. $\mathbf{p}^* = \mathbf{T}(\mathbf{p}^*)$; that is, a “fixed-point” of **T**

Fact

(Banach's)If $\mathbf{T} : S \to S$ *is a contraction in* $S \subset \Re^M$ *(that is,* $\exists r \in [0, 1)$ *such that* $\forall \mathbf{x}, \mathbf{y} \in S, \|T(\mathbf{x}) - T(\mathbf{y})\| \leq r \|\mathbf{x} - \mathbf{y}\|$ *) then* **T** *has a unique fixed-point, that can be found by successive approximation, irrespective of the starting point [3]*

- We identify conditions under which the power-adjustment transformation is a contraction.

Fixed points in $\Re$



The reverse triangle inequality

Fact

If the function $f : V \to \Re$ *satisfies* $f(\mathbf{x}+\mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$ *then* $|f(\mathbf{x}) - f(\mathbf{y})| \leq f(\mathbf{x}-\mathbf{y})$

Proof.

Without loss of generality, suppose that $f(\mathbf{x}) \geq f(\mathbf{y})$ which implies that $f(\mathbf{x}) - f(\mathbf{y}) \equiv |f(\mathbf{x}) - f(\mathbf{y})|$.

Observe that $\mathbf{x} \equiv (\mathbf{x}-\mathbf{y}) + \mathbf{y}$. By hypothesis, $f(\mathbf{x}) \equiv f((\mathbf{x}-\mathbf{y}) + \mathbf{y}) \leq f(\mathbf{x}-\mathbf{y}) + f(\mathbf{y})$ which implies that $f(\mathbf{x}) - f(\mathbf{y}) = |f(\mathbf{x}) - f(\mathbf{y})| \leq f(\mathbf{x}-\mathbf{y})$

□

Methodology: key argument

- Suppose g_i is quasi-semi-normal. Let $\|\mathbf{T}(\mathbf{x}) - \mathbf{T}(\mathbf{y})\| =$

$$\left\| \begin{bmatrix} g_1(\mathbf{x}) - g_1(\mathbf{y}) \\ \vdots \\ g_N(\mathbf{x}) - g_N(\mathbf{y}) \end{bmatrix} \right\| = \max \begin{bmatrix} |g_1(\mathbf{x}) - g_1(\mathbf{y})| \\ \vdots \\ |g_N(\mathbf{x}) - gf(\mathbf{y})| \end{bmatrix}$$

- $|g_i(\mathbf{x}) - g_i(\mathbf{y})| \leq g_i(\mathbf{x} - \mathbf{y})$ (reverse triangle ineq.)
- $\therefore \|\mathbf{T}(\mathbf{x}) - \mathbf{T}(\mathbf{y})\| \leq \max(g_1(\mathbf{x} - \mathbf{y}), \cdots, g_N(\mathbf{x} - \mathbf{y}))$
- Since g_i is both q-sub-homogeneous and q-non-decreasing, it follows that $g_i(\mathbf{x} - \mathbf{y}) \leq \|\mathbf{x} - \mathbf{y}\| \, g_i(\mathbf{1})$
- $\therefore \|\mathbf{T}(\mathbf{x}) - \mathbf{T}(\mathbf{y})\| \leq \|\mathbf{x} - \mathbf{y}\| \max(g_1(\mathbf{1}), \cdots, g_N(\mathbf{1})) = r \, \|\mathbf{x} - \mathbf{y}\|$ with $r := \max(g_1(\mathbf{1}), \cdots, g_N(\mathbf{1}))$
- If each $g_i(\mathbf{1}) < 1$, then $r \in [0, 1)$ and **T** *is* a contraction

Norms I

Let V be a vector space (see [5, pp. 11-12] for definition).

Definition

A function $f\colon\ V \to \Re$ is called a *semi-norm* on V, if it satisfies:

1. $f(v) \geq 0$ for all $v \in V$ (non-negativity)
2. $f(\lambda v) = |\lambda| \cdot f(v)$ for all $v \in V$ and all $\lambda \in \Re$ (homogeneity)
3. $f(v+w) \leq f(v) + f(w)$ for all $v, w \in V$ (*triangle ineq.*)

Definition

If f also satisfies $f(v) = 0 \iff v = \theta$ (where θ is the zero element of V), then f is called a *norm* and $f(v)$ is denoted as $||v||$

Norms II

Definition

The Hölder norm with parameter $p \geq 1$ ("p-norm") is denoted as $|| \cdot ||_p$ and defined for $x \in \Re^N$ as $\|\mathbf{x}\|_p = (|x_1|^p + \cdots + |x_N|^p)^{\frac{1}{p}}$

With $p = 2$, the Hölder norm becomes the familiar Euclidean norm. Also, $\lim_{p\to\infty} \|\mathbf{x}\|_p = \max(|x_1|, \cdots, |x_N|)$, thus:

Definition

For $x \in \Re^N$, the infinity or "max" norm is defined by $\|\mathbf{x}\|_\infty := \max(|x_1|, \cdots, |x_N|)$

Banach fixed-point theorem

Definition

A map T from a normed space $(V, \|\cdot\|)$ into itself is a *contraction* if there exists $r \in [0, 1)$ such that for all $x, y \in V$, $\|T(x) - T(y)\| \leq r\|x - y\|$

Theorem

(Banach' Contraction Mapping Principle) If T is a contraction mapping on V there is a unique $x^* \in V$ such that $x^* = T(x^*)$. Moreover, x^* can be obtained by successive approximation, starting from an arbitrary initial $x_0 \in V$. [3]

For Further Reading I

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