Abstract—A linear shift deterministic 3-way channel with reciprocal channel gains is considered in this work. The 3-way channel is an extension of the 2-way channel introduced by Shannon. Here, a number of six messages is exchanged, one message from each user to the two other users. Each user operates in a full-duplex mode. We derive the capacity region of this 3-way channel w.r.t. the linear shift deterministic channel model. To this end, an outer bound is derived using cutset and genie-aided upper bounds. Then, it is noted that the outer bound bears a resemblance to the capacity region of a related linear shift deterministic $Y$-channel. Utilizing a $\Delta$-$Y$ transformation, the optimal scheme for the related $Y$-channel is modified in a way that achieves the outer bound of the 3-way channel. Mainly, the capacity achieving communication schemes are based on multi-way relaying by signal alignment, interference neutralization and backward decoding. We also consider a scheme which is based on interference alignment only. It turns out, that for the symmetric linear deterministic 3-way channel, this scheme is optimal. Thus, backward decoding and the resulting delays are avoided.

I. INTRODUCTION

The linear deterministic channel model (LDCM) is a conceptual model to describe the influence of interference in a multi-user network as introduced in [1]. While noise is de-emphasized, the LDCM focuses on the impact of signal strength and superposition of multiple interfering signals. Capacity-achieving communication schemes derived in terms of the LDCM are useful tools to approximate the corresponding Gaussian channel capacity within a limited number of bits [2] at high signal-to-noise ratio. Furthermore, the principle of interference alignment (IA) [3] is also applicable for linear deterministic multi-user channels, as, e.g., shown in [4], [5]. A solid basis of works treating several different multi-user networking problems in terms of the linear deterministic channel model is already available, e.g., the sum capacity of the 2-user $X$-channel is treated in [5], and the sum-capacity of the $K$-user symmetric interference channel in [4].

The main focus of this paper is to characterize the capacity region of the linear deterministic 3-way channel. The 3-way channel is an extension of the two-way channel [6], a common setup in wireless communications where two users communicate with each other simultaneously over the same channel. This is clearly a very natural communication situation with 2 users. In the light of the LDCM, the capacity region of the two-way multiple access channel, the two-way broadcast channel and the two-way interference channel is discussed in [7].

The 3-way channel with perfect full-duplex operation is a basic extension of the two-way channel, in which all three users communicate with each other simultaneously. The reciprocal linear deterministic 3-way channel is depicted in Fig. 1. There is a total number of six messages involved. The sum-rate of the Gaussian 3-way channel has already been approximated within 2 bits in [8], recently. Note that a related conferencing 3-way channel is discussed in [9] for a finite-field channel model. But in that case, each transmitter has only one message that is dedicated for both receivers, i.e., three messages in total. The deterministic 3-way channel is similar to the deterministic 3-user $Y$-channel in [10], [11], [12], but it has no intermediate relay for the purpose of two-way relaying and signal alignment (SA) [13], as also discussed for MIMO channels in [14] and [15].

In our work presented here, we consider the capacity region of a linear deterministic 3-way channel (or $\Delta$-channel) as depicted in Fig. 1. For the 3-way channel, we first provide genie-aided upper bounds for the capacity region. Achieving this capacity region relies upon SA, interference neutralization (IN) [16] and backward decoding [17].

Our main tool is a $\Delta$-$Y$ transformation of the 3-way channel to an (extended) $Y$-channel, motivated by elementary electrical circuit theory [18], [19]. Furthermore, we integrate a capacity-achieving scheme for a symmetric 3-way channel that is purely based on signal-scale interference alignment (IA) as in [3]–[5]. For that case we also avoid backward decoding.
Organization. The reciprocal linear deterministic 3-way channel is introduced in Sect. II-A. Cut-set and genie-aided upper bounds on the capacity region are derived in Sect. II-B. We briefly describe the closely related Y-channel and its capacity region in Sect. III. In Sect. IV, the achievability of the capacity region of the 3-way channel is derived by means of the aforementioned $\Delta \gamma$ transformation. Furthermore, we discuss in Sect. V to what extent a purely IA based scheme can be used to achieve the capacity region. We conclude in Sect. VI.

II. 3-WAY CHANNEL

A. System Model: 3-Way Channel

In the 3-way channel, a user $T_i$ is a combined full-duplex transmitter $T_X_i$ and receiver $R_X_i$. We consider six independent messages $W_{ij}$ dedicated to be conveyed from $T_i$ to $T_j$ with $W_{ij} \in W_{ij} = \{1, \ldots, 2^{nR_{ij}}\}$, $R_{ij} \in \mathbb{R}^+$, for distinct $i,j \in \mathcal{K} := \{1, 2, 3\}$. The vector of all messages is denoted by:

$$w = (W_{12}, W_{21}, W_{13}, W_{31}, W_{23}, W_{32}).$$

(1)

The rate tuple $R$ and the total sum-rate $R_{\Sigma}$ are defined for rates $R_{ij} \in \mathbb{R}^+$ between $T_X_i$ and $R_X_j$ by:

$$R = (R_{12}, R_{21}, R_{13}, R_{31}, R_{23}, R_{32}),$$

$$R_{\Sigma} = R_{12} + R_{21} + R_{13} + R_{31} + R_{23} + R_{32}.$$  

(2)

(3)

$T_j$ encodes its messages into a codeword $^N x_j$. The $l$-th symbol of $^N x_j$ is an element of an alphabet $\mathcal{X}$ encoded as $x_j(l)$. The collections of messages, encoders, and decoders defines a code for the 3-way channel. Furthermore, rate tuple $R$ is called achievable if there is a sequence of codes such that the average error probability $\epsilon_N$ becomes arbitrarily small by increasing $N$. The set of all achievable rate tuples is the capacity region $C_{\Delta}$. In the LDCM, the physical channel between $T_i$ and $T_j$ is modelled by $n_{ij} \in \mathbb{N}$ bit pipes, and the transmitted symbols $x_j(l)$ are bit-vectors in $\mathcal{X} = \mathbb{F}_q$ with $q = \max_{i,j \in \mathcal{K}} \{n_{ij}\}$. The received signals $y_i$ at receivers $R_X_i$, $i \in \mathcal{K}$ are deterministic functions of the transmitted signals for distinct $i,j,k \in \mathcal{K}$:

$$y_i = S^{q-n_{ij}} x_j \oplus S^{q-n_{ik}} x_k,$$

(4)

where $S$ is a $q \times q$ lower shift matrix, having unit entries on the lower side-diagonal. The effect of noise is mimicked by clipping linearly shifted symbols. Note that loop-back self-interference is entirely cancelled from (4) due to the perfect full-duplex operation.

In a wireless channel, it is valid to assume reciprocity for the bidirectional links such that we may use the following parametrization $n_{ij} = n_{ji} = n_k$ holds for distinct $i,j,k \in \mathcal{K}$. We also assume:

$$n_3 \geq n_2 \geq n_1,$$  

(5)

as an ordering of parameters w.l.o.g., and obtain $n_3 = q$. We denote this reciprocal 3-way channel by $D3C(n_1, n_2, n_3)$.

B. Upper Bounds: Linear Deterministic 3-Way Channel

Cut-set upper bounds: The cut-set bounds of broadcast and multiple-access channels as in [1] state that users $T_i$ can not receive more bits than the number of incoming bit-levels and they can not transmit more bits than the number of outgoing bit-levels available:

$$R_{ij} + R_{ik} \leq \max \{n_k, n_j\},$$

$$R_{ji} + R_{ki} \leq \max \{n_k, n_j\},$$

(6)

(7)

for distinct $i,j,k \in \mathcal{K}$. These cut-set bounds already provide a loose upper bound $C_{\Delta, \text{cut-set}}$ on the actual capacity region $C_{\Delta}$. To obtain a tight capacity characterization, we include further genie-aided upper bounds similar to those derived in [10].

Genie-aided upper bounds: Receiver $T_1$ intends to decode the dedicated messages $W_{12}$ and $W_{13}$ using its received signal $^N y_1$ and its own messages $W_{21}, W_{31}$, with a reliable decoding strategy. Let the interfering message $W_{23}$ be provided to node $T_1$ as genie-aided side-information. Since $T_1$ already knows $W_{13}$ and $W_{23}$, it can reconstruct $^N x_3(1)$.

From $^N y_1$ and $^N x_3(1)$, $T_1$ can derive $^N x_2(1)$ from the deterministic function (4). $T_1$ then constructs $^N y_1$ from $^N x_1(1)$ and $^N x_2(1)$. With $W_{13}, W_{23}$ and $^N y_1$, $T_1$ can generate $^N x_2(2)$. These steps are repeated accordingly for all time instants from 2 to $N$ until $^N y_3$ is completely constructed. Therefore, by knowing $^N y_1, W_{21}, W_{31}$ and $W_{23}$ at $T_1$, it can reliably decode $W_{12}$ and $W_{13}$, and then reconstruct $^N y_3$ to reliably decode $W_{32}$. All messages are known at node $T_1$ now. Thus, for the genie-aided channel, any reliable code allows decoding $W_{32}$. From Fano’s inequality, we can derive:

$$N(R_{12} + R_{13} + R_{32}) \leq I(W_{12}, W_{13}, W_{32}; Y_1, W_{21}, W_{31}, W_{23}) + N\epsilon_N,$$

$$\leq H(Y_1) - H(Y_1 | w) + N\epsilon_N,$$

$$\leq H(Y_1) + N\epsilon_N,$$

$$\leq N(\max\{n_3, n_2\} + \epsilon_N) = N(n_3 + \epsilon_N),$$

where $\epsilon_N \to 0$ as $N \to \infty$. By letting $N \to \infty$, we get the bound $R_{12} + R_{13} + R_{32} \leq n_3$. Similar bounds can be derived by considering different receivers and side-information (cf. Appendix). By considering genie-aided and non-redundant cut-set bounds jointly, we obtain the following set of upper bounds on the capacity region $C_{\Delta}$:

$$R_{31} + R_{32} \leq n_2,$$

$$R_{13} + R_{23} \leq n_2,$$

(9)

(10)
Fig. 2. The reciprocal $Y$-channel with three transceivers $T_1, T_2$ and $T_3$ has six independent messages $W_j$, transmitted and six corresponding estimated messages $\tilde{W}_j$, received by the nodes, $i \neq j \in \mathcal{K}$. The channel gains are parameterized by $n_j \in \mathbb{N}$, for $j \in \mathcal{K}$.

\[ R_{12} + R_{13} + R_{32} \leq n_3, \]  \hspace{0.5cm} (11)
\[ R_{12} + R_{13} + R_{23} \leq n_3, \]  \hspace{0.5cm} (12)
\[ R_{21} + R_{23} + R_{13} \leq n_3 + n_2 - n_1, \]  \hspace{0.5cm} (13)
\[ R_{21} + R_{23} + R_{31} \leq n_3, \]  \hspace{0.5cm} (14)
\[ R_{31} + R_{32} + R_{21} \leq n_3, \]  \hspace{0.5cm} (15)
\[ R_{31} + R_{32} + R_{12} \leq n_3 + n_2 - n_1. \]  \hspace{0.5cm} (16)

The sum-capacity upper bound yields $R_{\Sigma} \leq 2n_3$. This set of bounds leads to the following lemma.

**Lemma 1.** The capacity region $C_{\Delta}$ of the D3C$(n_1, n_2, n_3)$ is outer bounded by $\bar{C}_{\Delta}$, i.e., $C_{\Delta} \subseteq \bar{C}_{\Delta}$, where:

\[ \bar{C}_{\Delta} = \{ R \in \mathbb{R}_+^9 | R \text{ satisfies (9)-(16)} \}. \]

This outer bound is in fact achievable. The achievability of this bound is proved via a $\Delta-Y$ transformation utilizing the optimal scheme for the $Y$-channel as a building block. Next, we briefly introduce the $Y$-channel.

### III. Y-Channel

Before we prove the achievability of Lemma 1, we briefly recapitulate the $Y$-channel whose capacity in terms of the LDCM is characterized in [10].

#### A. System Model: Linear Deterministic Y-Channel

The deterministic reciprocal $Y$-channel\(^2\) DYC$(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$ is depicted in Fig. 2. The definitions of the message vector, rate tuple, the transmission symbols and the encoding/decoding functions carry over from those given in Sect. II-A, but are denoted with the tilde notation to distinguish between the two models. In contrast to the 3-way channel, all users $T_j$ are connected via bidirectional reciprocal links to an intermediate relay $R$. The channel gain from $R$ to user $T_j$ is denoted by $\tilde{n}_j$. The gains are ordered w.r.t. \( \tilde{n}_1 \geq \tilde{n}_2 \geq \tilde{n}_3 \), \hspace{0.5cm} (17)

so that $q = \max_{i \in \mathcal{K}} \{ \tilde{n}_i \} = \tilde{n}_1$. Note that this ordering is reversed when comparing it with (5). The transmitted signals are vectors $x_j, x_R \in \mathbb{F}_q^2$ from $T_j$ and $R$, respectively.

\(^2\)Our notation slightly differs from [10] w.r.t. swapped indexation and tilde.

The received signal at $R$ and the received signals at $T_j$ are given by:

\[ y_R = \sum_{j=1}^{3} S_q^{-n_j} x_j, \]  \hspace{0.5cm} (18)
\[ y_j = S_q^{-n_j} x_R, \]  \hspace{0.5cm} (19)

respectively. Next, we re-state the capacity region of the linear shift deterministic $Y$-channel, which will be an essential part of the proof for the achievability of Lemma 1.

#### B. Capacity Region: Linear Deterministic Y-Channel

The capacity region $C_Y$ of the DYC$(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$ has already been characterized in [10], and is given by the set of rate tuples $R = (\tilde{R}_{12}, \tilde{R}_{13}, \tilde{R}_{31}, \tilde{R}_{23}, \tilde{R}_{32})$, satisfying:

\[ \tilde{R}_{12} + \tilde{R}_{13} + \tilde{R}_{32} \leq \tilde{n}_3, \]  \hspace{0.5cm} (20)
\[ \tilde{R}_{12} + \tilde{R}_{13} + \tilde{R}_{23} \leq \tilde{n}_3, \]  \hspace{0.5cm} (21)
\[ \tilde{R}_{12} + \tilde{R}_{13} + \tilde{R}_{32} \leq \tilde{n}_2, \]  \hspace{0.5cm} (22)
\[ \tilde{R}_{12} + \tilde{R}_{13} + \tilde{R}_{23} \leq \tilde{n}_3, \]  \hspace{0.5cm} (23)
\[ \tilde{R}_{21} + \tilde{R}_{23} + \tilde{R}_{13} \leq \tilde{n}_1, \]  \hspace{0.5cm} (24)
\[ \tilde{R}_{21} + \tilde{R}_{23} + \tilde{R}_{31} \leq \tilde{n}_2, \]  \hspace{0.5cm} (25)
\[ \tilde{R}_{31} + \tilde{R}_{32} + \tilde{R}_{21} \leq \tilde{n}_2, \]  \hspace{0.5cm} (26)
\[ \tilde{R}_{31} + \tilde{R}_{32} + \tilde{R}_{12} \leq \tilde{n}_1. \]  \hspace{0.5cm} (27)

There is an interesting resemblance between the bounds (20)-(27) and (9)-(16). This resemblance will be exploited to design an optimal scheme for the 3-way channel next.

### IV. $\Delta$-Y Transformation

Equating the upper bounds of the D3C$(n_1, n_2, n_3)$ in (9)-(16) and the DYC$(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$ in (20)-(27) yields:

\[ \tilde{n}_1 = n_2 + n_3 - n_1 \]  \hspace{0.5cm} (28)
\[ \tilde{n}_2 = n_3, \]  \hspace{0.5cm} (29)
\[ \tilde{n}_3 = n_2. \]  \hspace{0.5cm} (30)

In other words, the outer bound for the D3C$(n_1, n_2, n_3)$ coincides with the capacity region of a DYC$(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$. Note that the ordering of the channel gains in (5) and (17) still holds.

In order to show the achievability of the outer bound in Lemma 1, we first express the 3-way channel in terms of an extended $Y$-channel as depicted in Fig. 3. User $T_1$ is extended such that it behaves like a virtual relay $R$ which is also connected to a virtual user $\tilde{T}_1$ via an artificial...
sub-channel parametrized by $\tilde{n}_1$. At $R$, the topmost levels $\tilde{n}_2 + 1, \ldots, \tilde{n}_1$ are only accessible by $\tilde{T}_1$ and not visible for $T_2$ and $T_3$ (Fig. 4(a)) and hence only virtual within $T_1$. The residual link with $n_1$ from the previous D3C($n_1, n_2, n_3$) remains as a weak bidirectional link between $T_2$ and $T_3$ in the extended $Y$-channel eDYC($n_1, \tilde{n}_2, \tilde{n}_3, n_1$). The channel gain $n_1$ is still the weakest one, since:

$$n_1 \leq n_2 = \tilde{n}_3 \leq \tilde{n}_2 = n_3 \leq n_3 + n_2 - n_1 = \tilde{n}_1. \quad (31)$$

The optimal scheme for the $Y$-channel already achieves the outer bound $\mathcal{C}_D$ for $n_1 = 0$. However, in general we have $n_1 \geq 0$. Hence, we have to modify our scheme to deal with the additional interference over $n_1$.

A. Achievability of $\mathcal{C}_D$

Let all (virtual) users $\tilde{T}_1$, $T_2$, $T_3$ and $R$ apply the capacity-achieving SA scheme described in [10] as if it would be applied on a DYC($\tilde{n}_1, \tilde{n}_2, \tilde{n}_3$), but without decoding at the receivers yet. We consider $N$ uplink and $N$ downlink transmissions, over $N + 1$ time-instants. We call this scheme the 'original' scheme. For illustration, consider the uplink, downlink and the additional bidirectional link $n_1$ for the eDYC($\tilde{n}_1, \tilde{n}_2, \tilde{n}_3, n_1$) as depicted in Fig. 4(a).

In contrast to the original scheme for the DYC($\tilde{n}_1, \tilde{n}_2, \tilde{n}_3$), the scheme for the eDYC($\tilde{n}_1, \tilde{n}_2, \tilde{n}_3, n_1$) must be adapted to deal with the signals inherently transmitted over $n_1$. We will overlay the adapted scheme on top of the original scheme.

In particular, any signal transmitted by $T_i$, $i \in \{2, 3\}$, on the topmost levels $\tilde{n}_1 - n_1 + 1, \ldots, \tilde{n}_1$, will interfere at receiver $T_j$ on the lowermost levels $1, \ldots, n_1$. We discern three classes of interference over $n_1$ that are potentially received at $T_2$ and $T_3$ when applying the original scheme:

(a) The interference over $n_1$ received at $T_1$ is a dedicated signal from $T_j$ to $T_i$, which will also be forwarded from $R$ to $T_i$ in the next time-instant.

(b) The interference over $n_1$ received at $T_3$ is a dedicated signal from $T_2$ to $T_1$.

(c) The interference over $n_1$ received at $T_2$ is a dedicated signal from $T_3$ to $T_1$.

Class (a): To compensate the interference of class (a), we postpone decoding until the last signals of the $(N+1)$-th time-instant are received. The transmission scheme does not change w.r.t. the original one. Since the (uplink) transmitters of $T_2$ and $T_3$ are silent on the $(N+1)$-th time-instant, the (downlink) receivers of $T_2$ and $T_3$ receive no signal over $n_1$ at the $(N+1)$-th time-instant. Hence, $T_2$ and $T_3$ can decode their dedicated signals as received in the last hop. In fact, the downlink signals of the $(N+1)$-th time-instant behave analogously to the $(N+1)$-th hop of the original scheme. Since the class (a) interference of the $N$-th hop is a subset of the dedicated signals in the $(N+1)$-th hop, it is cancelled after decoding the dedicated signals of the $(N+1)$-th hop. With such a backward decoding scheme, the interference of class (a) is cancelled analogously for all preceding time-instants $N - 1, \ldots, 2, 1$.

Class (b): To compensate the interference of class (b), i.e., those bits received at $T_3$ over $n_1$ carrying a dedicated signal from $T_2$ to $T_1$, say $x_{12}$, we apply an IN scheme. In detail, $T_2$ pre-transmits the interference signal ($x_{12}(l)$) one time-instant in advance (in time-instant $l-1$) as follows. Assume that $T_3$ receives $[x_{R.3}(l), x_{R.3}(l)]^T$ from $R$ in the downlink at time-instant $l$, where $x_{R.3}(l)$ and $x_{R.3}(l)$ are binary vectors of lengths $n_1$ and $\tilde{n}_3 - n_1$, respectively (see Fig. 4(a)). Moreover, assume that $T_3$ receives interference from $x_{12}(l)$ over some bits of $x_{R.3}(l)$. To deal with this interference, $T_2$ pre-transmits $x_{12}(l)$ in time-instant $l-1$ in the uplink, over exactly the same levels where $x_{R.3}(l)$ is received in the uplink. By doing so, $T_3$ receives $x_{12}(l)$ twice over $x_{R.3}(l)$ in the downlink, once from $T_2$ and once from $R$. Since $x_{12}(l)$ is a binary vector, the addition of $x_{12}(l)$ to itself results in interference neutralization.

It remains to make sure that the pre-transmission does not disturb any other node. Clearly, $x_{12}$ does not disturb $T_2$ since it originates from the same node $T_2$. Also, $x_{12}$ does not disturb $T_1$ since $x_{12}$ is a desired signal at $T_1$, and thus the interfering $x_{12}$ is removed by backward decoding.

One more problem remains. Our approach using IN only works if $x_{R.3}(l)$ is received over levels that are accessible by $T_2$ in the uplink, i.e., the levels $1, \ldots, \tilde{n}_2$ at $R$. However, $x_{R.3}(l)$ might contain information from $T_1$, say $x_{31}$, which might not be accessible by $T_2$ in the uplink. This is exactly the case if $T_1$ sends $x_{31}$ over levels $\tilde{n}_2 + 1, \ldots, \tilde{n}_1$ at $R$ (blue area in Fig. 4(a)). However, the given problem can be solved easily by noting that the number of levels in the blue area in Fig. 4(a) is $\tilde{n}_1 - n_2$. We have $n_1 - \tilde{n}_2 = n_3 - n_1$ by (28), i.e., the same number of levels in the non-interfered downlink levels at $T_3$ (green area in Fig. 4(a)). Therefore, we exploit this interesting equality of the transformation: $R$ forwards $x_{31}$ over the non-interfered downlink levels at $T_3$ and the given problem is avoided. By pursuing such an approach, the impact of class (b) interference is completely eliminated.

Class (c): To compensate the interference of class (c) at $T_2$ received over $n_1$, i.e., a dedicated signal $x_{31}$ from $T_3$ to $T_1$, we apply a similar IN scheme. $T_3$ likewise additionally pre-transmits class (c) interference one time-instant in advance. $T_3$ can access $n_3$ levels in the uplink to $R$ that are potentially forwarded to $T_2$ in the downlink during the next time-instant.

However, the levels $\tilde{n}_3 + 1, \ldots, \tilde{n}_1$ (blue area in Fig. 4(b)) are not accessible by $T_3$ in the uplink. Thus, if the signal received by $T_2$ from $R$ in the downlink over levels $1, \ldots, n_3$ are sent over relay levels $\tilde{n}_3 + 1, \ldots, \tilde{n}_1$ in the uplink, then, $T_3$ can not perform IN. However this scenario can be avoided by sending all signals received in the uplink on the blue levels in Fig. 4(b), over the green levels in the downlink. This is possible since $\tilde{n}_2 - n_1 = n_1 - \tilde{n}_3$ holds by (28). In this case, these signals do not interfere with the levels $1, \ldots, n_1$ at $T_2$ which renders $T_3$ capable of performing IN.

For the downlink from $R$ to $T_3$, the pre-transmitted signals which are back-propagated to $T_3$ are known self-interference
Theorem 2. The capacity region \( \mathcal{C}_\Delta \) of the D3C\((n_1, n_2, n_3)\) is given by \( \mathcal{C}_\Delta \) defined in Lemma 3.

V. CAPACITY REGION OF THE SYMMETRIC CASE BY IA

Interestingly, signals conveyed over the weak link \( n_1 \) are not used for direct communication. The interfering signals over \( n_1 \) are cancelled or neutralized by the communication scheme proposed in Section IV-A, so that the impact of the link \( n_1 \) is effectively eliminated. A certain drawback of our previous scheme is that the receivers must wait for \( N + 1 \) time-instants to apply the backward decoding procedure. This is a very restrictive property, especially for delay-limited communications [20].

As a contrary approach, we now propose a purely IA-based communication scheme for the symmetric D3C\((m, m, m)\) that achieves the corresponding capacity region. The communication scheme for the D3C\((m, m, m)\) based on IA is proven with similar methods as the one in [10]. In this case, there is no need for backward decoding and interference neutralization.

Theorem 3. An interference alignment scheme based on bidirectional, cyclic and unidirectional communication suffices to achieve the outer bounds on the capacity region of a symmetric D3C\((m, m, m)\) with \( m \in \mathbb{N} \).

We consider a communication scheme of 3 components:

A) **Bidirectional**: For distinct \( i, j \in \mathcal{K} \), the pair of rates \( R_{ij}, R_{ji} \) is non-zero.

B) **Cyclic**: For distinct \( i, j, k \in \mathcal{K} \), the triple of rates \( R_{ij}, R_{jk}, R_{ki} \) is non-zero, whereas \( R_{ij} = R_{kj} = R_{ik} = 0 \).

C) **Unidirectional**: None of the above cases holds.

We now outline our proposed IA scheme based on the components A, B and C. We will begin with scheme A on the D3C\((m, m, m)\) operating at 2 bits per level. Pairs of users communicate bidirectionally. Then, we reduce the channel to D3C\((m', m', m')\) by removing the already used levels from scheme A. Next, scheme B with \( \frac{3}{2} \) bits per level is applied. Again we reduce the channel to D3C\((m'', m'', m'')\) removing the levels occupied by scheme B. In the last step, we apply scheme C allocating 1 bit per level. If the rate tuple to be achieved does not satisfy one of conditions A, B, and C, the corresponding scheme is merely discarded. We will show in the following that these schemes suffice to achieve the outer bounds of the capacity region for the D3C\((m, m, m)\).

A. **Bidirectional Communication on D3C\((m, m, m)\)**

We define the following three transmission parameters \( a, b, c \in \mathbb{N} \):

\[
a = \min\{R_{12}, R_{21}\}, \quad b = \min\{R_{13}, R_{31}\}, \quad c = \min\{R_{23}, R_{32}\}.
\]  

(32)
If $a = b = c = 0$ holds, scheme A is skipped and we continue with scheme B. We propose a signal allocation such that 2 bits per level are achieved. The signals are $x_{12}, x_{31} \in F_2$, $x_{31}, x_{13} \in F_2$, and $x_{32}, x_{23} \in F_2$. To transmit these signals, $a + b + c \leq m$ levels are allocated as depicted in Fig. 5. The interference signals $x_{ji}$ and $x_{ij}$ are aligned at $T_k$ with pairwise distinct $i, j, k \in \mathcal{K}$.

This allocation scheme is only feasible if enough levels are available at the transmitters and receivers for all bidirectional streams. For $R \in \mathcal{C}$, the following must hold on $a, b, c$:

$$a + b + c \leq R_{12} + R_{13} + R_{32} \leq m.$$  \hfill (33)

This is also true for all other upper bounds. For the yet unused levels, we still need to achieve the residual rate-vector:

$$R' = (R_{12} - a, R_{21} - a, R_{13} - b, R_{31} - b, R_{23} - c, R_{32} - c) = (R'_{12}, R'_{21}, R'_{13}, R'_{31}, R'_{23}, R'_{32}).$$  \hfill (34)

So far, at least three components will already be zero due to the min-expressions in (32). We remove the allocated levels so that the reduced $\mathcal{D}3\mathcal{C}(m', m', m')$ is parameterized by:

$$m' = m - a - b - c.$$  \hfill (35)

Clearly, the reduced channel remains symmetric.

B. Cyclic Communication on $\mathcal{D}3\mathcal{C}(m', m', m')$

Given that the conditions for scheme B hold, and depending on the residual rate-vector $R'$ computed in (34), we apply either clock-wise cyclic communication $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ with parameter $d$ or counter-clock-wise cyclic communication $1 \rightarrow 3 \rightarrow 2 \rightarrow 1$ with parameter $e$. The parameters $d, e \in \mathbb{N}$ are:

$$d = \min(R'_{21}, R'_{13}, R'_{32}), \quad e = \min(R'_{12}, R'_{31}, R'_{23}).$$  \hfill (36)

Note that either $d$ or $e$ must be zero, since bidirectional communication is already taken care of by the previous scheme A. The definitions in (36) provide two cases:

$$d > 0 \Rightarrow e = 0, \quad a = R_{12}, \quad b = R_{31}, \quad c = R_{23},$$  \hfill (37)

$$e > 0 \Rightarrow d = 0, \quad a = R_{21}, \quad b = R_{13}, \quad c = R_{32}.$$  \hfill (38)

If both $d = e = 0$, this section is skipped and we continue with scheme C. In the following, we propose a signal allocation scheme such that $\frac{3}{2}$ bits per level are achieved.

For case (37), $R'_{12} = R'_{31} = R'_{23} = 0$ and $R'_{21}, R'_{13}, R'_{32}$ are non-zero. The signals are $x_{21}, x_{13}, x_{32} \in F_2$, allocated in blocks of $d$ levels as depicted in Fig. 6. The constraint $2d \leq m'$ must hold at each user to provide a feasible allocation. Signal $x_{32}$ is transmitted by $T_2$ on both intervals of size $d$ and $T_1$ applies interference cancellation to decode $x_{13}$ from $x_{13} + x_{32}$. The interference signals $x_{21}$ and $x_{32}$ are aligned at $T_3$.

Sufficiently many levels are available for scheme B on the reduced $\mathcal{D}3\mathcal{C}(m'', m'', m'')$ if $R \in \mathcal{C}$, since:

$$2d \leq R'_{12} + R'_{32} \leq R_{13} + R_{32} - b - c \leq m - R_{12} - b - c = m - a - b - c,$$  \hfill (39)

The second case (38) for counter clock-wise communication is derived analogously, but with the indices swapped and with an adapted allocation (cf. Fig. 7). In particular, we have $R'_{21} = R'_{13} = R'_{32} = 0$ and $R'_{12}, R'_{31}, R'_{23}$ are non-zero. In analogy to (39), the allocations for counter-clock-wise communication with parameter $e$ satisfies $R \in \mathcal{C}$:

$$2e \leq R'_{31} + R'_{23} \leq m - a - b - c = m'.$$  \hfill (40)

For the yet unused levels, the residual rate-vector is:

$$R'' = (R'_{12} - d, R'_{21} - e, R'_{13} - e, R'_{31} - d, R'_{23} - d, R'_{32} - e) = (R''_{12}, R''_{21}, R''_{13}, R''_{31}, R''_{23}, R''_{32}),$$  \hfill (41)

over the $\mathcal{D}3\mathcal{C}(m'', m'', m'')$ (where either $d$ or $e$ is zero) with parameter:

$$m'' = m' - 2d - 2e.$$  \hfill (42)
C. Unidirectional Communication on D3C(m′′, m′′, m′′)
Six possible non-zero rate tuples remain that are not yet covered by the previous schemes A and B:

\[
\begin{align*}
R_{12}^\prime, R_{31}^\prime, R_{32}^\prime &\neq 0, (R_{12}^\prime, R_{31}^\prime, R_{23}^\prime) \neq 0, \\
(R_{12}^\prime, R_{13}^\prime, R_{23}^\prime) &\neq 0, (R_{12}^\prime, R_{13}^\prime, R_{32}^\prime) \neq 0, \\
(R_{12}^\prime, R_{13}^\prime, R_{23}^\prime) &\neq 0, (R_{23}^\prime, R_{13}^\prime, R_{32}^\prime) \neq 0.
\end{align*}
\]

These cases pairwise exclude each other. W.l.o.g., we only consider the unidirectional case \((R_{12}^\prime, R_{13}^\prime, R_{23}^\prime) \neq 0\), here. The remaining cases are derived by analogous steps. We have \(R_{31}^\prime = R_{32}^\prime = R_{12}^\prime = 0\) and we parameterize the three non-zero rates by \(f, g, h \in \mathbb{N}\):

\[
R_{12}^\prime = f, R_{13}^\prime = g, R_{23}^\prime = h. \tag{42}
\]

The signals are \(x_{12} \in \mathbb{F}_2^f, x_{13} \in \mathbb{F}_2^g, x_{23} \in \mathbb{F}_2^h\). A number of \(f + g + h \leq m^{\prime\prime}\) levels are allocated as depicted in Fig. 8.

![Fig. 8. C) Allocation of signals to bit-levels in unidirectional communication over D3C(m′′, m′′, m′′).](image)

Since we demand \(R^\prime \in \mathcal{C}\), we discern two cases depending on the previous scheme B. In the first case, we assume that clock-wise cyclic communication was applied before. Recall that either \(d\) or \(e\) must be zero. If \(d = 0\) and \(e = 0\), then:

\[
f + g + h = R_{12}^\prime + R_{13}^\prime + R_{23}^\prime = R_{12}^\prime + R_{13}^\prime + R_{23}^\prime - 2d
\leq \left(\frac{\text{(3)}}{9}\right) R_{12}^\prime + R_{13}^\prime + R_{23}^\prime - a - b - c - 2d
\leq m - a - b - c - 2d \stackrel{\text{(3)}}{=} m - 2d. \tag{43}
\]

Otherwise, if \(e > 0\) (counter clock-wise) and \(d = 0\), then:

\[
f + g + h \stackrel{\text{(3)}}{=} R_{12}^\prime + R_{13}^\prime + R_{23}^\prime - a - b - c - e
\leq \left(\frac{\text{(3)}}{14}\right) R_{12}^\prime + R_{13}^\prime + R_{23}^\prime - a - b - e
\leq m - (e + c) - a - b - e = m - 2e. \tag{44}
\]

For (44), we use \(e \leq R_{23}^\prime = R_{23} - c\) from (42) and (34). Since either \(d\) or \(e\) is zero, combining (43) and (44) yields:

\[
f + g + h \leq m - 2d - 2e \stackrel{\text{(4)}}{=} m^{\prime\prime}. \tag{45}
\]

The remaining steps, showing that each corner point of the capacity region is achievable is analogous to [10, Thms. 3&4] and omitted here.

VI. CONCLUSIONS
We have studied the capacity region of the linear shift deterministic 3-way channel. Tight upper bounds on the capacity region are characterized using cut-set upper bounds and genie-aided upper bounds. Our main result is that the reciprocal linear deterministic 3-way channel can be transformed into an extended linear deterministic Y-channel with an additional weak link. Then, we apply the capacity-achieving signal alignment scheme proposed by Chaaban et al. for the reciprocal linear deterministic Y-channel and further extend it with an interference neutralization and backward decoding procedure. As a result, we show that it suffices to achieve the capacity region by relaying all messages over the user with the strongest incident sub-channels using a network-coded signal-alignment scheme.

As a complementary approach, we have also proposed an interference alignment scheme that achieves the capacity region of a fully symmetric sub-channel of the 3-way channel without resorting to relaying, backward decoding and interference neutralization. For an extension of this work, it would be interesting to see whether similar transformations can be applied to transform centralized networks to decentralized ones, and vice versa.

APPENDIX A
GENIE-AIDED UPPER BOUNDS
The remaining upper bounds on the capacity region of the D3C\((n_1, n_2, n_3)\) are derived similar to Section II-B. We only discuss the main differences in this appendix.

(i): To derive the bound \(R_{12} + R_{13} + R_{23} \leq n_3\) we provide \(W_{32}\) as side-information to the receiver of \(T_1\), and proceed similar to Section II-B. That is, we prove that the enhanced \(T_1\) can construct \(y_1^N\) from which it can decode \(W_{32}\).

(ii): To derive the bound \(R_{21} + R_{23} + R_{13} \leq n_3 + n_2 - n_1\), we provide \(W_{31}\) and \(x_3^N\) to \(T_2\) as side-information, where \(\bar{x}_3^N\) denotes the lowermost \(n_2 - n_1\) bits of \(x_3^N\). Providing these bits is necessary since \(T_2\) can only obtain the topmost \(n_1\) symbols of \(x_3^N\) from \(y_2^1\) and \(x_1^1\) after decoding \(W_{21}\) (recall that given \(W_{31}\) and \(W_{21}\), \(T_2\) can construct \(x_1^1\)). By combining the topmost \(n_1\) bits of \(x_3^N\) and the lowermost \(n_2 - n_1\) bits provided by the side-information, \(T_2\) can construct \(y_1^1\) and hence also \(x_1^2\), since it knows \(W_{31}\) from side-information and \(W_{21}\) after decoding. Similarly, all components of \(y_2^N\) can be constructed, and \(W_{13}\) can be decoded. Thus, by Fano’s inequality, we can write:

\[
N(R_{21} + R_{23} + R_{13}) \leq I(W_{21}, W_{23}, W_{13}; y_2^N, \bar{x}_3^N, x_1^N, W_{12}, W_{32}, W_{31}) + N\epsilon_N
\leq H(y_2^N, \bar{x}_3^N) - H(y_2^N, \bar{x}_3^N|w) + N\epsilon_N
\leq H(y_2^N, \bar{x}_3^N) + N\epsilon_N
\leq N(\max\{n_3 + n_2 - n_1, n_1 + n_2 - n_1\} + \epsilon_N)
= N(n_3 + n_2 - n_1 + \epsilon_N),
\]
where $\epsilon_N \to 0$ as $N \to \infty$. This provides the upper bound $R_{21} + R_{23} + R_{13} \leq n_3 + n_2 - n_1$ after dividing by $N$ and letting $N \to \infty$.

(iii): To derive $R_{21} + R_{23} + R_{31} \leq n_3$, we provide $W_{13}$ to $T_2$ and proceed similar to (i), by showing that $T_2$ can construct $y_N^2$ and decode $W_{31}$.

(iv): To derive $R_{31} + R_{32} + R_{21} \leq n_3$, we give $\hat{x}_1^N$ and $W_{12}$ to $T_3$ as side-information, where $\hat{x}_1^N$ denotes the lowermost $n_3 - n_2$ bits of $x_1^N$. By proceeding similar to (ii), we can show that $T_3$ can construct $y_N^2$ given this side-information, and then decode $W_{21}$, leading to the desired bound.

(v): To derive the bound $R_{31} + R_{32} + R_{12} \leq n_3 + n_2 - n_1$, we give $W_{21}$ and $\hat{x}_2^N$ to $T_3$, where $\hat{x}_2^N$ denote the lowermost $n_3 - n_1$ bits of $x_2^N$. Similar to (ii), $T_3$ is able to construct $y_N^1$ given this side-information, and then to decode $W_{12}$, leading to the desired bound.

As a result, we obtain the upper bounds on capacity region as given by (9) to (16).

REFERENCES


