Abstract—We investigate the power allocation problem in distributed sensor networks that are used for target object classification. In the classification process, the absence, the presence, or the type of a target object is observed by the sensor nodes independently. Since these local observations are noisy and thus unreliable, they are fused together as a single reliable observation at a fusion center. The fusion center uses the best linear unbiased estimator in order to accurately estimate the reflection coefficient of target objects. We utilize the average deviation between the estimated and the actual reflection coefficient as a metric for defining the objective function. First, we demonstrate that the corresponding optimization of the power allocation leads to a signomial program which is in general quite hard to solve. Nonetheless, by using the proposed system model, fusion rule and objective function, we are able to optimize the power allocation analytically and can hence present a closed-form solution. Since the power consumption of the entire network may be limited in various aspects, three different cases of power constraints are discussed and compared with each other. In addition, a sensitivity analysis of the optimal power allocation with respect to perfect and imperfect parameter knowledge is worked out.

Index Terms—Analytical power allocation, energy-efficient optimization, distributed target classification, network resource management, information fusion.

I. INTRODUCTION

In this paper, we investigate the power allocation problem in distributed sensor networks that are used for active radar applications. Each sensor node (SN) individually and independently emits a radar signal and receives the reflected echo from a jointly observed target object. The particular observations serve as classification feature in order to classify the type of the present target object. Since the local observations at each SN are noisy and thus unreliable, they are combined into a single reliable observation at a remotely located fusion center to increase the overall system performance. In the classification process, the absence, the presence, and the type of the present target object are distinguished. The fusion center uses the best linear unbiased estimator in order to accurately estimate the reflection coefficient of the present target object, where each object is assumed to be uniquely characterized by its reflection coefficient. This setup is illustrated in Figure 1, whose technical components will be specified in detail later.

The research on distributed detection was originated from the attempt to combine signals of different radar devices [1]. Currently, distributed detection is rather discussed in the context of wireless sensor networks, where the sensor units may also be radar nodes [2]–[4]. In [5], the power allocation problem for distributed wireless sensor networks, which perform object detection and classification, is only treated for ultra-wide bandwidth (UWB) technology. Other applications, which require or benefit from detection and classification capabilities, are localization and tracking [6] or through-wall surveillance [7]. In [8], an approximate solution of the power allocation problem is proposed, which allows for an analytical treatment of output power-range limitation per sensor node. The optimal power allocation in passive radar systems, instead of active systems, is investigated in [9]. For active radar systems an optimal solution to the power allocation problem is only known for high signal-to-noise ratios (SNRs), see [10]. However, a closed-form optimal solution to the power allocation problem has not yet been investigated in the context of object classification for the whole range of SNR. The main difficulty for optimizing the power consumption is associated with finding a closed-form equation for the overall classification probability. As example, for the Bayesian hypothesis test criterion the overall classification probability cannot be analytically evaluated [11]. This limits the usability of this criterion for solving the power allocation problem. Bounds, such as the Bhattacharyya bound [12], are also difficult to use for optimizing multidimensional problems. Hence, the best power allocation scheme is still an open problem in order to improve the overall classification probability.

In the present work, we analytically optimize the power allocation and hence can present a closed-form solution for
a network of amplify-and-forward SNs. Based on a simple system model, we apply a linear fusion rule and utilize the average deviation between the estimated and the actual reflection coefficient as a metric for defining the objective function. This approach is the key idea in the present work which enables the analytical optimization of the power allocation in closed-form. Since the power consumption of the entire network may be limited in various aspects, three different cases of power constraints are discussed and compared with each other. They lead to explicit policies for the optimal power allocation. Furthermore, we demonstrate that all considered constraints lead to signomial optimization problems which are in general quite hard to solve. In addition, we simulatively analyze the sensitivity of the optimal objective with respect to perfect parameter knowledge and subsequently with respect to imperfect channel state information. These are the main contributions of the current work.

The present work is organized as follows. We start with a description of the underlying technical system in the next section. Subsequently, the power allocation problem is specified and analytically solved. The achieved results are then discussed and carefully compared with each other. Finally, simulative results are presented and their behavior is described.

Mathematical Notations:
Throughout this paper we denote the sets of natural, integer, real, and complex numbers by \( \mathbb{N}, \mathbb{Z}, \mathbb{R}, \) and \( \mathbb{C} \), respectively. The imaginary unit is denoted by \( i \). Note that the set of natural numbers does not include the element zero. Moreover, \( \mathbb{R}_+ \) denotes the set of non-negative real numbers. Furthermore, we use the subset \( \mathbb{F}_N \subseteq \mathbb{N} \) which is defined as \( \mathbb{F}_N := \{1, \ldots, N\} \) for any given natural number \( N \). We denote the absolute value of a real or complex-valued number \( z \) by \( |z| \) while the expected value of a random variable \( v \) is denoted by \( \mathbb{E}[v] \). Moreover, the notation \( V^* \) stands for the optimal value of an optimization variable \( V \) at an optimum point of the corresponding optimization problem.

II. OVERVIEW AND TECHNICAL SYSTEM DESCRIPTION
At any instance of time, a network of \( K \in \mathbb{N} \) independent and spatially distributed SNs receives random observations. If a target object is present, then the received power at the SN \( S_k \) is a part of its own emitted power which is back-reflected from the jointly observed target object and is weighted by its reflection coefficient \( r_k \). The object may be of \( I \) different types. It should be noted that sheer detection may be treated as the special case of \( I = 2 \) which corresponds to the decision ‘some object is present’ versus ‘there is no object’. We assume that all different object types and their corresponding reflection coefficients are known by the network. Moreover, the received signal at each SN is weighted by the corresponding channel coefficient and disturbed by additive noise. It is obvious that the sensing channel is wireless. The sensing task and its corresponding communication task for a single classification process are performed in consecutive time slots. All SNs take samples from the disturbed received signal and amplify them without any additional data processing in each time slot.

The amplified samples remain buffered in the SNs during the current time slot. Simultaneously in the same time slot, new radio waves are emitted by all SNs for the next observation and classification process. In addition, the buffered samples of the former classification process are communicated to the fusion center which is placed in a remote location. We assume that SNs have only limited sum-power available for sensing the object and communicating to the fusion center. Furthermore, each SN may be limited in its transmission power-range due to transmission-power regulation standards or due to the functional range of its circuit elements. The sensing task as well as the communication to the fusion center are performed by using distinct waveforms (pulse shapes) for each SN so as to distinguish sensing and communication of different SNs. Each waveform has to be suitably chosen in order to suppress inter-user (inter-node) interference at other SNs and also at the fusion center. Furthermore, we assume that in the frequency domain each waveform is orthogonal to all other waveforms in order to calculate the sensing power of each SN independent from its communication power. Hence, the \( K \) received signals at the fusion center are uncorrelated and assumed to be conditionally independent. Each received signal at the fusion center is influenced by the corresponding channel coefficient and additive noise, as well. The communication channel between the SNs and the fusion center can either be wireless or wired. The disturbed received signals at the fusion center are weighted and combined together in order to obtain a single reliable observation \( \tilde{r} \) of the actual reflection coefficient \( r_k \). Note that we disregard time delays within all transmissions and assume synchronized data communication.

In the following subsections, we mathematically describe the underlying system model that is depicted in Figure 2. The continuous-time system is modeled by its discrete-time equivalent, where the sampling rate of the corresponding signals is equal to the target observation rate, for the sake of simplicity.

A. Target Object
We assume that all objects have the same size, shape and alignment, but different material and, hence, complex-valued
reflection coefficients \( r_i \in \mathbb{C}, \ i \in F_i \). Thus, the reflection coefficients are the only recognition features in this work. The a-priori probability of occurrence for each object type is denoted by \( \pi_i \in \mathbb{R}_+ \), \( i \in F_i \), with \( \sum_{i} \pi_i = 1 \). The root mean squared value of the reflection coefficients is given as

\[
    r_{\text{rms}} := \sqrt{\sum_{i \in F_i} \pi_i |r_i|^2}.
\]

Furthermore, the actual target object is assumed to be static during consecutive observation steps.

### B. Sensing channel

Each propagation path of the sensing channel, from each SN to the object and again back to the same SN, is described by a corresponding random channel coefficient \( g_k \). For the investigation of the power allocation problem, the concrete realization of channel coefficients is needed and hence can be used for postprocessing of the received signals at the SNs. We assume that all channel coefficients are complex-valued and static during each target observation step. Furthermore, the coherence time of sensing channels is assumed to be much longer than the whole length of the classification process. Thus, the expected value and the quadratic mean of each coefficient during each observation step can be assumed to be equal to their instantaneous values, i.e., \( \mathbb{E}[g_k] = g_k \) and \( \mathbb{E}[|g_k|^2] = |g_k|^2 \). In practice, it is often difficult to measure or estimate these coefficients. Thus, the results of the present work are applicable for scenarios where all channel coefficients can somehow be accurately estimated during each observation process or they are nearly deterministic and thus can be measured before starting the radar task.

Furthermore, all channel coefficients are assumed to be uncorrelated and jointly independent. Note that each channel coefficient includes the radar cross section, the influence of the antenna, the impact of the filters, as well as all additional attenuation of the target signal.

At the input of each SN, the disturbance is modeled by the complex-valued additive white Gaussian noise (AWGN) \( m_k \) with zero mean and finite variance \( M_0 := \mathbb{E}[|m_k|^2] \) for all \( k \). Note that the channel coefficient and the noise on the same propagation path are also uncorrelated and jointly independent.

### C. Sensor nodes

We model each SN by an amplify-and-forward unit with extended capabilities, where both sensing and communication signals are transmitted simultaneously. The sensing signal \( w_k \), without loss of generality, is assumed to be non-negative, real-valued and deterministic. The expected value of its sensing power is then described by

\[
    W_k := \mathbb{E}[|w_k|^2] = |w_k|^2, \ k \in F_K.
\]

Note that the specific value of \( w_k \) is adjustable and will be determined later by the power allocation procedure.

The ratio of the communication signal to the received signal is described by the non-negative real-valued amplification factor \( u_k \) which is assumed to be constant over the whole bandwidth and power-range. Thus, the communication signal and the expected value of its communication power are described by

\[
    x_k := (r_ig_k w_k + m_k)u_k, \ k \in F_K
\]

and

\[
    X_k := \mathbb{E}[|x_k|^2] = (r_{\text{rms}}^2 |g_k|^2 W_k + M_0)u_k^2, \ k \in F_K,
\]

respectively. The amplification factor is an adjustable parameter and will be determined later by the power allocation procedure, as well. Note that the instantaneous power fluctuates from observation to observation depending on the present target object.

If the received signal is negligible in comparison to the output signal and if the nodes have smart power components with low-power dissipation loss, then the average power consumption of each node is approximately equal to its average output power \( W_k + X_k \). The addition of both transmission powers is justified because the corresponding signals are assumed to be separated by distinct waveforms. We also assume that the output power-range of each SN is limited by \( P_{\text{max}} \) and that the average power consumption of all SNs together is limited by the sum-power constraint \( P_{\text{tot}} \). Hence, the constraints

\[
    W_k + X_k \leq P_{\text{max}}
\]

\[
    \iff (1 + r_{\text{rms}}^2 |g_k|^2 u_k^2)W_k + M_0 u_k^2 \leq P_{\text{max}}, \ k \in F_K \quad (5)
\]

and

\[
    \sum_{k \in F_K} \left( W_k + X_k \right) \leq P_{\text{tot}}
\]

\[
    \iff \sum_{k \in F_K} \left( (1 + r_{\text{rms}}^2 |g_k|^2 u_k^2)W_k + M_0 u_k^2 \right) \leq P_{\text{tot}} \quad (6)
\]

arise consequently. We remark that the described method can also be extended to individual output power-range constraints per SN.

Note that the sum-power constraint \( P_{\text{tot}} \) is a reasonable approach to compare energy-efficient radar systems.

### D. Communication channel

Analogous to the sensing channel, each propagation path of the communication channel is described by a corresponding random channel coefficient \( h_k \). But in contrast to the sensing channel, the concrete realization of each communication channel coefficient is measurable by using pilot sequences at each SN. Accordingly, the channel coefficients can be used for postprocessing of received signals at the fusion center. We assume that all channel coefficients are complex-valued and static during each target observation step. Furthermore, the coherence time of communication channels is assumed to be much longer than the whole length of the classification process. Thus, the expected value and the quadratic mean of each channel coefficient can be assumed to be equal to their instantaneous values, i.e., \( \mathbb{E}[h_k] = h_k \) and \( \mathbb{E}[|h_k|^2] = |h_k|^2 \).

Furthermore, all channel coefficients are assumed to be uncorrelated and jointly independent. Note that each channel
coefficient includes the influence of the antenna, the impact of the filters, as well as all additional attenuation of the corresponding sensor signal.

At the input of the fusion center, the disturbance on each communication path is modeled by the complex-valued AWGN $n_k$ with zero mean and finite variance $N_0 := \mathcal{E}[|n_k|^2]$ for all $k$. Note that the channel coefficient and the noise on the same propagation path are also uncorrelated and jointly independent.

E. Fusion center

The fusion center combines the different local observations into a single reliable one by applying a linear combiner. Thus, the received signals are weighted with the complex-valued factors $v_k$ and summed up to yield an estimate $\tilde{r}$ of the actual target signal $r_i$. In this way, we obtain

$$ y_k := (x_k h_k + n_k) v_k, \quad k \in F_K, \quad (7) $$

and hence,

$$ \tilde{r} := \sum_{k \in F_K} y_k \sum_{k \in F_K} w_k g_k u_k h_k v_k + \sum_{k \in F_K} (m_k u_k h_k + n_k) v_k. \quad (8) $$

Note that each weight can be written as $v_k = |v_k| \exp(j \vartheta_k)$, $k \in F_K$, where $\vartheta_k$ is a real-value number which represents the phase of the corresponding weight.

Note that the fusion center can separate the input streams because the communication channel is either wired or the data communication is performed by distinct waveforms for each SN. Consequently, if the communication channel is wireless then a matched-filter bank is essential at the input of the fusion center to separate the data streams of different SNs. In addition, we do not consider inter-user (inter-node) interferences at the fusion center because of the distinct waveform choices.

In order to obtain a single reliable observation at the fusion center, the value $\tilde{r}$ should be a good estimate for the present reflection coefficient $r_i$. Thus, we optimize the sensing power $W_k$, the amplification factors $u_k$, and the weights $v_k$ in order to minimize the average absolute deviation between $\tilde{r}$ and the true reflection coefficient $r_i$. This optimization procedure is elaborately explained in the next section. After determining $W_k$, $u_k$, and $v_k$, the fusion center observes a disturbed version of the true reflection coefficient $r_i$ at the input of its decision unit. Hence, by using the present system model, we are able to separate the power allocation problem from the classification problem and optimize both independently.

F. Some remarks on the system model

All described assumptions are necessary to obtain a framework suitable for analyzing the power allocation problem, without studying detection, classification and estimation problems in specific systems and their settings.

The accurate estimation of all channel coefficients is necessary for both the radar process and the power allocation. Sometimes it is not possible to estimate the transmission channels; consequently the channel coefficients $g_k$ and $h_k$ remain unknown. In such cases, the radar usually fails to perform its task.

Since the channel coefficients $g_k$ are in practice difficult to estimate or to determine, our approach rather shows theoretical aspects of the power allocation than the practical realization and implementation. Hence, the presented results act as theoretical bounds and references for comparing real radar systems.

Moreover, since the coherence time of communication channels as well as sensing channels is assumed to be much longer than the whole length of the classification process, the proposed power allocation method is applicable only for scenarios with slow-fading channels.

Note that only the linear fusion rule together with the proposed objective function enable optimizing the power allocation in closed-form. The optimization of power allocation in other cases is in general hardly amenable analytically.

In general, SNs have only one power amplifier and a single antenna. The antenna is usually connected to a circulator in order to separate the signal of the transmitter to the antenna from the signal of the antenna to the receiver, which is not depicted in Figure 2. The power amplifier is also shared for sensing and communication tasks, but not considered in this work.

In order to increase the available power-range at each SN, time-division multiple-access (TDMA) can be used to completely separate the sensing task from the communication task and perform each task in a different time slot.

The introduced system model describes a baseband communication system without considering time, phase and frequency synchronization problems.

In order to distinguish the current operating mode of each SN in what follows, we say a SN is inactive or idle if the allocated power is zero. We say a SN is active if its allocated power is positive. Finally, we say a SN is saturated if the limitation of its output power-range is equal to the allocated power, i.e., $P_{\text{max}} = W_k + X_k$.

An overview of all notations that we will use hereinafter and are needed for the description of each observation process is depicted in Table I.

### Table I

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
<td>number of all nodes;</td>
</tr>
<tr>
<td>$F_K$</td>
<td>the index-set of $K$ nodes;</td>
</tr>
<tr>
<td>$K$</td>
<td>number of all active nodes;</td>
</tr>
<tr>
<td>$I$</td>
<td>number of different reflection coefficients;</td>
</tr>
<tr>
<td>$r_i$</td>
<td>reflection coefficient of $i$th target object;</td>
</tr>
<tr>
<td>$r_{\text{max}}$</td>
<td>root mean squared absolute value of reflection coefficients;</td>
</tr>
<tr>
<td>$g_k$, $h_k$</td>
<td>complex-valued channel coefficients;</td>
</tr>
<tr>
<td>$m_k$, $n_k$</td>
<td>complex-valued zero-mean AWGN;</td>
</tr>
<tr>
<td>$M_0$, $N_0$</td>
<td>variances of $m_k$ and $n_k$;</td>
</tr>
<tr>
<td>$u_k$, $v_k$</td>
<td>non-negative amplification factors and complex-valued weights;</td>
</tr>
<tr>
<td>$\vartheta_k$</td>
<td>phase of $v_k$;</td>
</tr>
<tr>
<td>$\phi_k$</td>
<td>phase of the product $g_k \cdot h_k$;</td>
</tr>
<tr>
<td>$W_k$, $X_k$</td>
<td>sensing and communication signal of $k$th sensor node;</td>
</tr>
<tr>
<td>$y_k$</td>
<td>input signals of the combiner;</td>
</tr>
<tr>
<td>$P_{\text{max}}$</td>
<td>output power-range constraint of each sensor node;</td>
</tr>
<tr>
<td>$P_{\text{tot}}$</td>
<td>sum-power constraint.</td>
</tr>
</tbody>
</table>
III. POWER ALLOCATION

In this section, we introduce the power optimization problem and consecutively present its analytical solutions for different power constraints. First, we investigate the case where only a sum-power constraint \( P_{\text{max}} \in \mathbb{R}_+ \) for the cumulative sum of the expected power consumption of each SN is given. Afterwards, we present the analytical solution of the power allocation problem for the case where the average transmission power of each SN is limited by the output power-range limitation \( P_{\text{max}} \in \mathbb{R}_+ \). Finally, we extend the power allocation problem to the case where both constraints simultaneously hold and present the corresponding optimal solution.

In general, the objective is to maximize the overall classification probability, however, a direct solution to the allocation problem does not exist, since no analytical expression for the overall classification probability is available. Instead, we minimize the average deviation between \( \hat{r} \) and \( r_i \), in order to determine the power allocation. The motivation for this method is the separation of the power allocation problem from the object classification procedure, as described in the last section. The corresponding optimization problem is elaborately described in the next subsection.

A. The optimization problem

As mentioned in the last section, the value \( \hat{r} \) should be a good estimate for the actual reflection coefficient \( r_k \) of the present target object. In particular, we aim at finding estimators \( \hat{r} \) of minimum mean squared error in the class of unbiased estimators for each \( i \).

The estimate \( \hat{r} \) is unbiased simultaneously for each \( i \) if \( \mathcal{E}[\hat{r} - r_i] = 0 \), i.e., from equation (8) with (2) we obtain the identity

\[
\sum_{k \in \mathcal{F}_K} \sqrt{W_k} g_k u_k h_k v_k = 1.
\]

(9)

This identity is our first constraint in what follows. Note that the mean of the second sum in (8) vanishes since the noise is zero-mean. Furthermore, we do not consider the impact of both random variables \( g_k \) and \( h_k \) as well as their estimates in our calculations because the coherence time of both channels is assumed to be much longer than the target observation time. Note that equation (9) is complex-valued and may be separated as

\[
\sum_{k \in \mathcal{F}_K} \sqrt{W_k} u_k |v_k g_k h_k| \cos(\theta_k + \phi_k) = 1
\]

(10)

and

\[
\sum_{k \in \mathcal{F}_K} \sqrt{W_k} u_k |v_k g_k h_k| \sin(\theta_k + \phi_k) = 0,
\]

(11)

where \( \theta_k \) and \( \phi_k \) are phases of \( v_k \) and \( g_k h_k \), respectively.

The objective is to minimize the mean squared error \( \mathcal{E}[|\hat{r} - r_i|^2] \). By using equation (8) and the identity (9) we may write the objective function as

\[
V := \mathcal{E}[|\hat{r} - r_i|^2] = \sum_{k \in \mathcal{F}_K} |v_k|^2 (u_k^2 |h_k|^2 M_0 + N_0), \tag{12}
\]

Note that (12) is only valid if \( m_k \) and \( n_k \) are white and jointly independent.

As mentioned in the last section, each SN has an output power-range limitation and the expected overall power consumption is also limited. Hence, the objective function is also subject to (5) and (6), which are our second and last constraints, respectively.

In summary, the optimization problem is to minimize the mean squared error in (12) with respect to \( u_k, v_k, \) and \( W_k \), subject to constraints (5), (6), (10) and (11).

Note that each resulting optimization problem is a signomial program, which is a generalization of geometric programming, and is thus non-convex in general [13]. Furthermore, it is important to note that signomial programs cannot be transformed in general into convex optimization problems [14]. Since we have not found any transformation from signomials into corresponding convex programs thus far, we hence apply the general method of Lagrangian multipliers with equality constraints to solve all optimization problems in the present work, see [15, pp. 321–358] and [16, pp. 275–368]. In order to ensure global optimality of our results, we consecutively show the following four steps during each solving procedure. First, we relax each problem into an optimization problem with extended ranges for all involved variables to ensure an optimization within the interior-set. Second, all stationary points of the associated Lagrangian are localized by considering the corresponding derivatives to obtain necessary conditions. Third, we show that the number of stationary points is equal to one, which indirectly implies that the considered stationary point is also a regular point for all (active) constraints. Finally, to obtain a sufficient condition, we then show that the stationary point has a convex neighborhood which corresponds with a minimum. In summary, the applied method is based on the regularity of all active constraints as well as first and second order optimality conditions which together guarantee for global optimality. At this point, we emphasize that obtaining global optimality is similarly achievable by applying vector space methods [17], using interval analysis [18], or utilizing a proper constraint qualification (CQ) together with Karush-Kuhn-Tucker conditions (KKT) [19].

B. Power allocation subject to the sum-power constraint

In this case, the output power-range constraint per SN is assumed to be greater than the sum-power constraint and thus does not have any effect on the optimization problem, because the feasible set of the optimization problem is only limited by the sum-power constraint. This leads to the corresponding constrained Lagrange function (relaxation with respect to the range of \( W_k, u_k \) and \( |v_k| \))

\[
L_1(W_k, u_k, v_k; \eta_1, \eta_2, \tau; \xi) := \sum_{k \in \mathcal{F}_K} |v_k|^2 (u_k^2 |h_k|^2 M_0 + N_0) + \left(1 - \sum_{k \in \mathcal{F}_K} \sqrt{W_k} u_k |v_k g_k h_k| \cos(\theta_k + \phi_k)\right) \eta_1
\]

\[
- \left(\sum_{k \in \mathcal{F}_K} \sqrt{W_k} u_k |v_k g_k h_k| \sin(\theta_k + \phi_k)\right) \eta_2
\]

\[
+ P_{\text{tot}} - \xi - \sum_{k \in \mathcal{F}_K} (1 + r_{\text{rms}}^2 |g_k|^2 u_k^2) W_k + M_0 u_k^2 \tau,
\]
where \( \eta_1, \eta_2 \) and \( \tau \) are Lagrange multipliers while \( \xi \) is a slack variable.

In order to satisfy (11), all phases \( \vartheta_k + \phi_k \) have to be equal to \( q_k \pi, q_k \in \mathbb{Z} \), for all \( k \in \mathcal{K} \). If there were a better solution for \( \vartheta_k + \phi_k \), then the first partial derivatives of \( L_1 \) with respect to \( \vartheta_k \) would vanish at that solution, due to the continuity of trigonometric functions. But the first derivatives would lead to the equations \( \eta_1 \sin(\vartheta_k + \phi_k) = \eta_2 \cos(\vartheta_k + \phi_k) \) which cannot simultaneously satisfy both equations (10) and (11) for all \( \eta_1 \) and \( \eta_2 \). Thus, \( q_k \pi \) is the unique solution. Hence, we may consequently write a modified Lagrange function as
\[
\tilde{L}_1(W_k, u_k, |v_k|, q_k; \eta_1, \tau; \xi) := \sum_{k \in \mathcal{K}} |v_k|^2 (u_k^2 |h_k|^2 M_0 + N_0) + \left(1 - \sum_{k \in \mathcal{K}} \sqrt{W_k u_k} |v_k g_k h_k| \cos(q_k \pi)\right) \eta_1 + \left(P_{\text{tot}} - \xi - \sum_{k \in \mathcal{K}} (1 + r_{\text{rms}}^2 g_k^2 u_k^2) W_k + M_0 u_k^2 \right) \tau .
\]
(14)

At any stationary point of \( \tilde{L}_1 \) the first partial derivatives of \( \tilde{L}_1 \) with respect to \( W_k, u_k, |v_k|, \eta_1 \) and \( \tau \) must vanish, if they exist. For all \( l \in \mathcal{K} \), this leads to
\[
\frac{\partial \tilde{L}_1}{\partial W_l} = \frac{-u_l |v_l h_l g_l| \cos(q_l \pi)}{2 \sqrt{W_l}} \eta_l - (1 + r_{\text{rms}}^2 |g_l|^2 u_l^2) \tau = 0 ,
\]
(15)
\[
\frac{\partial \tilde{L}_1}{\partial |v_l|} = 2 |v_l| (u_l^2 |h_l|^2 M_0 + N_0) - \sqrt{W_l u_l} |v_l h_l g_l| \cos(q_l \pi) \eta_l = 0 ,
\]
(16)
\[
\frac{\partial \tilde{L}_1}{\partial u_l} = 2 |v_l|^2 u_l |h_l|^2 M_0 - \sqrt{W_l} |v_l h_l g_l| \cos(q_l \pi) \eta_l - 2 u_l (W l r_{\text{rms}}^2 |g_l|^2 + M_0) \tau = 0 ,
\]
(17)
\[
\frac{\partial \tilde{L}_1}{\partial \eta_l} = 1 - \sum_{k \in \mathcal{K}} \sqrt{W_k u_k} |v_k g_k h_k| \cos(q_k \pi) = 0
\]
(18)
and
\[
\frac{\partial \tilde{L}_1}{\partial \tau} = P_{\text{tot}} - \xi - \sum_{k \in \mathcal{K}} (1 + r_{\text{rms}}^2 g_k^2 u_k^2) W_k + M_0 u_k^2 \tau = 0 .
\]
(19)

By multiplying (16) with \( |v_l| \), summing up the outcome over all \( l \), and using the identities (10) and (12), we obtain
\[
\eta_1 = 2 V
\]
(20)
which is a positive real number due to definition of \( V \). Because of the last relationship and according to (16), the value of \( \cos(q_l \pi) \) must be a positive number and hence each \( q_l \) must be an even integer number. Thus, we can choose \( q_l^* = 0 \) for all \( l \in \mathcal{K} \). and conclude
\[
\vartheta_l^* = -\phi_l , \ l \in \mathcal{K} .
\]
(21)
This solution gives the identity \( \cos(q_l^* \pi) = 1 \) which can be incorporated into (15), (16), (17) and (18).

Again by multiplying (16) with
\[
\frac{1}{2} u_l |h_l g_l| \sqrt{W_l} \quad \text{and summing up the outcome over all } l , \ \text{and using (10), (12) and (20), we obtain}
\]
\[
V = \frac{\eta_1}{2} = \left[ \sum_{k \in \mathcal{K}} \frac{u_l^2 |h_l g_k|^2 W_k}{u_l^2 |h_l|^2 M_0 + N_0} \right]^{-1} .
\]
(23)

In turn, by incorporating (23) into (16), it yields
\[
|v_l| = \frac{V u_l |h_l g_l| \sqrt{W_l}}{u_l^2 |h_l|^2 M_0 + N_0} .
\]
(24)
for all \( l \in \mathcal{K} \).

Note that for each feasible \( u_l \) and \( W_l, l \in \mathcal{K} \), equation (24) describes a feasible value for each \( |v_l| \). Since for each \( u_l W_l > 0 \) the relation \( |v_l| > 0 \) consequently follows, the feasible optimal values of each \( |v_l| > 0 \) are not on the boundary \( |v_l| = 0 \). Thus, finding optimal values for each \( u_l \) and \( W_l, l \in \mathcal{K} \), leads to optimum values for each \( |v_l|, l \in \mathcal{K} \), due to the convexity of (14) with respect to each \( |v_l| \). Hence, finding a unique global optimum for \( u_l \) and \( W_l, l \in \mathcal{K} \), yields the sufficient condition for the globally optimal solution of the minimization problem (14).

We replace each \( |v_l| \) in (15) and (17) with (24) and thus we obtain two equations for \( \tau \) as
\[
\tau = \frac{-V^2 u_l^2 |g_l h_l|^2}{(1 + u_l^2 r_{\text{rms}}^2 |g_l|^2) (u_l^2 |h_l|^2 M_0 + N_0)}
\]
(25)
and
\[
\tau = \frac{-V^2 W_l |g_l h_l|^2 N_0}{(W l r_{\text{rms}}^2 |g_l|^2 + M_0) (u_l^2 |h_l|^2 M_0 + N_0)^2} .
\]
(26)

Note that because of the negativity of \( \tau \), due to (25) or (26), and positivity of \( \eta_1 \), the Lagrangian (14) is also convex in both \( u_l \) and \( W_l, l \in \mathcal{K} \), as well. Hence, this Lagrangian is convex near any optimum/stationary point in each \( u_l, |v_l| \) and \( W_l \), but it seems not to be a jointly convex function, at all. Since this Lagrangian is separately convex in each direction, a possible stationary point can never be a maximum. As the derivatives in (15)–(19) are linear or at most quadratic separately for each variable, we will see later that only a single feasible stationary point exists.

For the sake of simplicity and in order to compare the results later on, we define new quantities as
\[
\alpha_k := \frac{M_0}{r_{\text{rms}}^2 |g_k|^2} \quad \Rightarrow \quad \alpha_k \in \mathbb{R}_+ ,
\]
(27)
\[
\beta_k := \frac{N_0}{|h_k|^2} \quad \Rightarrow \quad \beta_k \in \mathbb{R}_+ ,
\]
(28)
and
\[
\bar{u}_k := M_0 u_k^2 \Leftrightarrow u_k = \sqrt{\frac{\bar{u}_k}{M_0}} .
\]
(29)

By direct algebra from (25) and (26), we infer
\[
W_l = \frac{\bar{u}_l \alpha_l (\bar{u}_l + \beta_l)}{\alpha_l \beta_l - \bar{u}_l^2} , \ l \in \mathcal{K} .
\]
(30)
To satisfy the positivity of each $W_i$, the inequality
\[ \alpha_i \beta_i > \tilde{u}_i^2, \quad l \in \mathbb{F}_K, \]
must hold, which will be used later. By using (27)–(30), we may rewrite (19) and (23) as
\[ 1 = (P_{tot} - \xi) \left[ \sum_{k \in \mathbb{F}_K} \frac{\tilde{u}_k^2 (\alpha_k + \beta_k) + 2\tilde{u}_k \alpha_k \beta_k}{\alpha_k \beta_k - \tilde{u}_k^2} \right]^{-1} \]  
(32)
and
\[ V^{-1} = \frac{1}{r_{rms}} \sum_{k \in \mathbb{F}_K} \frac{\tilde{u}_k^2}{\alpha_k \beta_k - \tilde{u}_k^2} \]  
(33)
respectively. In turn, we incorporate (32) into (33) and infer
\[ V^{-1} = \frac{P_{tot} - \xi}{r_{rms} \sum_{k \in \mathbb{F}_K} \gamma_k} \sum_{k \in \mathbb{F}_K} \frac{\gamma_k \tilde{u}_k}{\alpha_k \beta_k + \beta_k} + 2\alpha_k \beta_k. \]  
(34)
As is well-known, the arithmetic mean in (34) is less than its greatest element such that the inequality
\[ V^{-1} \leq \frac{P_{tot} - \xi}{r_{rms} \max_{k \in \mathbb{F}_K} \left\{ \frac{\tilde{u}_k}{\alpha_k (\alpha_k + \beta_k) + 2\alpha_k \beta_k} \right\}} \]
(35)
arises consequently. It is obvious that (35) is strictly decreasing with respect to $\xi$. Thus, the optimal value for the slack variable is zero, i.e., $\xi^* = 0$. In (35), equality holds, if and only if, some elements are zero and all other ones are equal. In addition, it is obvious that (35) is strictly increasing in each $\tilde{u}_k$ and in turn the maximum value of a certain $\tilde{u}_k$ is achieved if for all $l \neq k, l \in \mathbb{F}_K$, the identity $\tilde{u}_l = 0$ holds, since the sum-power is kept constant. This means that only one SN is active and all other ones are idle. Hence, we can calculate the value of the corresponding $\tilde{u}_k$ from (32) as
\[ \tilde{u}_k = \sqrt{\frac{(\alpha_k \beta_k + P_{tot})^2 + \alpha_k \beta_k P_{tot}}{\alpha_k \beta_k + P_{tot}}} - \frac{\alpha_k \beta_k P_{tot}}{\alpha_k \beta_k + P_{tot}}. \]  
(36)
This value can be incorporated into (35) to obtain
\[ V^{-1} = \frac{P_{tot}^2}{r_{rms} \sum_{k \in \mathbb{F}_K} \gamma_k} \max_{k \in \mathbb{F}_K} \left\{ \frac{1}{c_k^2(P_{tot}) - P_{tot}^2} \right\} \]  
(37)
with the disturbance-intensity
\[ c_k(P) := \sqrt{\alpha_k \beta_k + \sqrt{(\alpha_k + P)(\beta_k + P)}}, \quad k \in \mathbb{F}_K, P \in \mathbb{R}_+. \]  
(38)
The value of $V^{-1}$ is maximal if the disturbance-intensity $c_k(P_{tot})$ is minimal. Hence, we re-index all SNs such that the inequality
\[ c_k(P_{tot}) \leq c_{k+1}(P_{tot}), \quad k \in \mathbb{F}_K - 1, \]  
(39)
holds and with that only the first SN is active, even if the first few disturbance-intensities are equal. From (24), (29), (30),
\[ V^* = \frac{\nu_{rms}^2}{P_{tot}^2} (\nu_1^2(P_{tot}) - P_{tot}^2), \]  
(40)
\[ \hat{u}_1^* = \sqrt{\frac{\alpha_1 \beta_1}{\alpha_1 + \beta_1 + P_{tot}} + \frac{\alpha_1 \beta_1 P_{tot}}{\alpha_1 + \beta_1 + P_{tot}}}, \]  
(41)
\[ u_1^* = \sqrt{\frac{\nu_1^*}{P_{tot}}}, \]  
(42)
\[ W_1^* = \frac{\alpha_1 \beta_1}{\alpha_1 + \beta_1 + P_{tot}} - \frac{1}{2}, \]  
(43)
\[ |v_1^*| = \frac{V^* \hat{u}_1^*}{r_{rms} |h_1|}, \]  
(44)
and
\[ |v_k^*| = W_k^* = u_k^* = 0, \quad k \in \mathbb{F}_K, k \neq 1. \]  
(45)
As mentioned before, the Lagrangian (14) is separately convex in each direction, such that possible stationary points cannot be maximum ones. On the other hand, as shown in (40)–(45), there exists only a single feasible solution for the set of derivatives in (15)–(19). This means that only a single stationary point exists which cannot be a maximum. To be a saddle point is also not possible, because then there would at least exist one additional stationary point which is not the case here. Thus, the Lagrangian (14) must actually be a jointly convex function in the neighborhood of its stationary point. Furthermore, since the number of stationary points is equal to one, all equality (active) constraints are regular. Hence, the separate convexity together with the regularity condition is even a sufficient condition for global optimality in the present case, see also [15]. Note also that all above results are the solution of the relaxed Lagrangian (13) with extended range of all variables, and nevertheless this solution coincides with the original range of all variables.

Note that by using the above results, the corresponding fusion rule is simplified by discarding the influence of inactive SNs from the fusion rule. The fusion rule (8) becomes
\[ \hat{\gamma} = y_1 = r_i + (m_1 h_1 u_1^* + n_1) v_1^*, \quad i \in \mathbb{F}_I. \]  
(46)
The equations (40)–(45) and (21) are the optimal solutions of the power allocation problem only subject to the sum-power constraint. They are hence the main contribution of the present subsection.

C. Power allocation subject to individual power constraints

In the current case, the sum-power constraint is assumed to be much greater than the output power-range constraint and thus does not have any effect on the optimization problem, because the feasible set of the optimization problem is only limited by the output power-range constraints. This leads to the corresponding constrained Lagrange function (relaxation

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1We give the name disturbance-intensity to $c_k$ because it behaves intrinsically like a metric for noise. Since $1/c_k$ also describes the quality of the $k$th SN, it can be interpreted as its reliability, see [20].
with respect to the range of $W_k$, $u_k$ and $|v_k|$

$$L_2(W_k, u_k, v_k; \eta_1, \eta_2, \lambda_k; q_k)$$

$$:= \sum_{k \in F_K} |v_k|^2 (u_k^2 |h_k|^2 M_0 + N_0)
+ \left(1 - \sum_{k \in F_K} \sqrt{W_k} u_k |v_k g_k h_k| \cos(\theta_k + \phi_k)\right) \eta_1
- \left(\sum_{k \in F_K} \sqrt{W_k} u_k |v_k g_k h_k| \sin(\theta_k + \phi_k)\right) \eta_2
+ \sum_{k \in F_K} \left(P_{\text{max}} - q_k - (1 + r_{\text{rms}}^2 |g_k|^2 u_k^2) W_k - M_0 u_k^2\right) \lambda_k,$$

(47)

where $\lambda_k$ are new Lagrange multipliers while $q_k$ are new slack variables.

Since the behavior of $L_2$ is identical to that of $L_1$ with respect to $|v_k|$ and $\theta_k$, we obtain the same results for the phases as given in (21). Hence, we may modify $L_2$ as

$$\tilde{L}_2(W_k, u_k, |v_k|; \eta_1, \lambda_k; q_k)$$

$$:= \sum_{k \in F_K} |v_k|^2 (u_k^2 |h_k|^2 M_0 + N_0)
+ \left(1 - \sum_{k \in F_K} \sqrt{W_k} u_k |v_k g_k h_k|\right) \eta_1
+ \sum_{k \in F_K} \left(P_{\text{max}} - q_k - (1 + r_{\text{rms}}^2 |g_k|^2 u_k^2) W_k - M_0 u_k^2\right) \lambda_k.$$  

(48)

Note that since the equality $\sin(\theta_k^* + \phi_k) = 0$ holds due to (21), the constraint (11) is discarded in (48).

At any stationary point of $\tilde{L}_2$ the first partial derivatives of $\tilde{L}_2$ with respect to $W_k$, $u_k$, $|v_k|$, $\eta_1$ and $\lambda_k$ must vanish, if they exist. For all $l \in F_K$, this leads to

$$\frac{\partial \tilde{L}_2}{\partial W_l} = -\frac{u_l |v_l h_l g_l|}{2 \sqrt{W_l}} \eta_1 - (1 + u_l^2 r_{\text{rms}}^2 |g_l|^2) \lambda_l = 0,$$

(49)

$$\frac{\partial \tilde{L}_2}{\partial |v_l|} = 2 |v_l| (u_l^2 |h_l|^2 M_0 + N_0) - \sqrt{W_l} u_l |h_l g_l| \eta_1 = 0,$$

(50)

$$\frac{\partial \tilde{L}_2}{\partial u_l} = 2 |v_l|^2 u_l |h_l|^2 M_0 - \sqrt{W_l} u_l |h_l g_l| \eta_1 - 2 u_l (W_l r_{\text{rms}}^2 |g_l|^2 + M_0) \lambda_l = 0,$$

(51)

$$\frac{\partial \tilde{L}_2}{\partial \eta_1} = 1 - \sum_{k \in F_K} \sqrt{W_k} u_k |v_k g_k h_k| = 0$$

(52)

and

$$\frac{\partial \tilde{L}_2}{\partial \lambda_l} = P_{\text{max}} - q_l - (1 + r_{\text{rms}}^2 |g_l|^2 u_l^2) W_l - M_0 u_l^2 = 0.$$  

(53)

By similar procedure as described in Subsection III-B, we obtain the same results as given in (23), (24) and (30), because the equations (49)–(52) and (15)–(18) are pairwise the same except of the difference between $\tau$ and $\lambda_l$. On the one hand, incorporating $W_l$ from (53) into (23), and using the same definition as in (27)–(29), lead to

$$V^{-1} = \frac{1}{r_{\text{rms}}^2 \sum_{k \in F_K} (\tilde{u}_k (P_{\text{max}} - q_k - \tilde{u}_k) - \tilde{u}_k)}$$

(54)

which is obviously strictly decreasing with respect to each $q_k$. Thus, the optimal value for each slack variable is zero, i.e., $q_k^* = 0$ for all $k \in F_K$. On the other hand, comparing $W_l$ from (53) with (30), leads to

$$\tilde{u}_k = \sqrt{\frac{(\alpha_k \beta_k + P_{\text{max}})}{\alpha_k + \beta_k + P_{\text{max}}}} - \frac{\alpha_k \beta_k}{\alpha_k + \beta_k + P_{\text{max}}}, k \in F_K.$$  

(55)

Since equation (54) is strictly increasing in the number $K$ of SNs and (55) holds for all SNs, we infer that all SNs are active. From (24), (29), (30), (38) and (55) we thus conclude

$$V^* = \left[\frac{P_{\text{max}}^2}{r_{\text{rms}}^2 \sum_{k \in F_K} \frac{1}{\tilde{u}_k^2}}\right]^{-1}$$

(56)

$$u_k^* = \sqrt{\frac{u_k^*}{M_0}}, k \in F_K,$$

(57)

$$W_k^* = \frac{\tilde{u}_k^* \alpha_k (u_k^* + \beta_k)}{\alpha_k \beta_k - (u_k^*)^2}, k \in F_K,$$

(58)

and

$$|v_k|^2 = \frac{V^* u_k^*}{r_{\text{rms}}^2 |h_k|^2 \sqrt{\tilde{u}_k^* + \beta_k \sqrt{\alpha_k \beta_k - (u_k^*)^2}, k \in F_K}}.$$  

(59)

As mentioned in Subsection III-B, the global optimality of the obtained results is also trivially reasoned, first because of the optimization of the relaxed Lagrange function (47) with extended range of all variables, second since the global optimum point of the relaxed problem coincides with the original range of all variables, and finally there exists only a single stationary point which has a jointly convex neighborhood corresponding to a minimum and satisfying the regularity condition.

Note that by using the above results, the corresponding fusion rule cannot be simplified, since all SNs are active and they cannot thus be discarded from the fusion rule.

The equations (55)–(59) and (21) are the optimal solution of the power allocation problem only subject to the output power-range constraint per SN. They are hence the main contribution of the present subsection.

D. Comparison of the solutions

As we have shown in Subsection III-B, the SN with the smallest $c_k(P_{\text{tot}})$ consumes the whole available sum-power $P_{\text{tot}}$, because the combination of its sensing and communication channel is the best compared to other SNs. All other SNs do not get any transmission power, since their information reliability is too poor to be considered for data fusion. They can be discarded from the fusion rule such that the observation of the target object is less interfered by noise and consequently results in a better data communication. Note that the information reliability of each SN is only determined by the value of its corresponding $c_k(P_{\text{tot}})$.

In contrast, if the transmission power of each SN is individually limited and no sum-power constraint is given, then
all SNs are active and their transmission power is equal to the output power-range constraint $P_{\text{max}}$, according to (53). In order to compare both methods from Subsection III-B and III-C, the values in (40) and (56) are needed. Note that for a fair comparison of both allocation methods in a certain scenario, an equal overall power is necessary, i.e., $P_{\text{tot}} = KP_{\text{max}}$.

Note that $\tilde{r}$ is an unbiased estimator for each $r_i$ due to constraint (9). By similar methods we can also minimize the mean squared error in both cases without restricting ourself to unbiased estimators. Obviously, the optimal value of $V$ will then be smaller than that in (40) or (56).

### E. Power allocation subject to both types of constraints

In the current subsection, we consider the optimization problem from Subsection III-A subject to all constraints, i.e., sum-power constraint as well as output power-range constraint per SN. Two of three different cases can be singled out and reduced to preceding instances.

First, if $KP_{\text{max}} < P_{\text{tot}}$, then the sum-power constraint is irrelevant, because the feasible set is only limited by the output power-range constraints. Hence, the power allocation problem reduces to the one described in Subsection III-C with results given in (55)–(59) and (21). The only difference is that a part of the available sum-power remains unallocated and cannot be used.

Secondly, if $P_{\text{tot}} \leq P_{\text{max}}$, then the output power-range constraints are irrelevant, because the feasible set is only limited by the sum-power constraint. Hence, the power allocation problem is equal to the one described in Subsection III-B. The corresponding results are described by (40)–(45) and (21).

The case of $P_{\text{max}} < P_{\text{tot}} \leq KP_{\text{max}}$ is most challenging. The amount of the available sum-power is possibly inadequate to supply all SNs with power $P_{\text{max}}$. Besides, it is not possible to allocate the available sum-power only to a single SN since $P_{\text{max}} < P_{\text{tot}}$. Hence, it will be shown that for the optimal solution only a subset of $K \leq K, K > 1$, SNs are active. Similar to the procedures in the previous subsections, we consider the corresponding constrained Lagrange function (relaxation with respect to the range of $W_k, u_k$ and $|v_k|$)

$$L_3(W_k, u_k, v_k; \eta_1, \eta_2, \tau, \lambda_k; \xi, \varphi_k)$$

$$:= \sum_{k \in F_K} |v_k|^2 (u_k^2|h_k|^2M_0 + N_0)$$

$$+ \left(1 - \sum_{k \in F_K} \sqrt{W_k}u_k |v_kg_kh_k| \cos(\theta_k + \varphi_k)\right)\eta_1$$

$$- \left(\sum_{k \in F_K} \sqrt{W_k}u_k |v_kg_kh_k| \sin(\theta_k + \varphi_k)\right)\eta_2$$

$$+ \left(P_{\text{tot}} - \xi - \sum_{k \in F_K} (1 + r_{\text{rms}}^2|g_k|^2u_k^2)W_k + M_0u_k^2\right)\tau$$

$$+ \sum_{k \in F_K} \left(P_{\text{max}} - \varphi_k - (1 + r_{\text{rms}}^2|g_k|^2u_k^2)W_k - M_0u_k^2\right)\lambda_k.$$  

(60)

Since the behavior of $L_3$ is identical to that of $L_1$ and $L_2$ with respect to $|v_k|$ and $\varphi_k$, we obtain the same results for the phases as given in (21). Hence, we may modify $L_3$ as

$$\tilde{L}_3(W_k, u_k, v_k; \eta_1, \tau, \lambda_k; \xi, \varphi_k)$$

$$:= \sum_{k \in F_K} |v_k|^2 (u_k^2|h_k|^2M_0 + N_0)$$

$$+ \left(1 - \sum_{k \in F_K} \sqrt{W_k}u_k |v_kg_kh_k| \right)\eta_1$$

$$+ \left(P_{\text{tot}} - \xi - \sum_{k \in F_K} (1 + r_{\text{rms}}^2|g_k|^2u_k^2)W_k + M_0u_k^2\right)\tau$$

$$+ \sum_{k \in F_K} \left(P_{\text{max}} - \varphi_k - (1 + r_{\text{rms}}^2|g_k|^2u_k^2)W_k - M_0u_k^2\right)\lambda_k.$$  

(61)

Note that since the equality $\sin(\varphi_k + \varphi_k) = 0$ holds due to (21), the constraint (11) is discarded in (61).

At any stationary point of $\tilde{L}_3$ the first partial derivatives of $\tilde{L}_3$ with respect to $W_k, u_k, v_k, \eta_1, \eta_2$ and $\lambda_k$ must vanish, if they exist. For all $l \in F_K$, this leads to

$$\frac{\partial \tilde{L}_3}{\partial W_l} = 0,$$

$$\frac{\partial \tilde{L}_3}{\partial u_l} = 0,$$

$$\frac{\partial \tilde{L}_3}{\partial \eta_1} = 0,$$

$$\frac{\partial \tilde{L}_3}{\partial \eta_2} = 0,$$

$$\frac{\partial \tilde{L}_3}{\partial \lambda_l} = 0.$$  

(62)-(66)

By the same method as described in Subsection III-B, we obtain the same results as given in (23), (24) and (30), because the equations (62)–(66) and (15)–(19) are pairwise the same except of the difference between $\tau$ and $\tau + \lambda_t$. According to (67), we are able to calculate the powers $W_l$ in terms of $\varphi_l$ and $u_l$. By using the same definition as in (27)–(29) and incorporating (30), (38) and (67) into (23) and (66), we derive

$$V^{-1} = \frac{1}{r_{\text{rms}}^2} \sum_{k \in F_K} \frac{1}{\left(\frac{c_k(P_{\text{max}} - \varphi_k)}{r_{\text{max}}^2 - \varphi_k}\right)^2 - 1}$$

(68)

and

$$P_{\text{tot}} = \xi - \sum_{k \in F_K} (P_{\text{max}} - \varphi_k).$$

(69)

As one can see, the minimization of the signomial program in (60) is reduced to the maximization of (68) subject to (69) and $0 \leq \varphi_k \leq P_{\text{max}} < P_{\text{tot}}$ for all $k \in F_K$ with respect to each $\varphi_k$. Since the new maximization problem is of special structure, it is amenable to an optimal solution with the aid of
Then, we can conclude

\[ \tilde{f} = \tilde{f}_{\text{SNs}} = \tilde{f}_{\text{SNs}} \]

which is obviously negative for all positive \( P \). Thus, \( c_k(P) \) is strictly decreasing in \( P \), and in turn, the objective in (68) is strictly decreasing in each \( \varrho_k \). To show the convexity more is needed. Since each element of the series (68) is equal to

\[ \frac{1}{(c_k(P))} = \frac{(\sqrt{\alpha_k\beta_k} - \sqrt{(\alpha_k + P)\beta_k + P})^2 - P^2}{(\alpha_k - \beta_k)^2}, \]

the second derivative of each element is given by

\[ \frac{\sqrt{\alpha_k\beta_k}}{2\sqrt{(\alpha_k + P)\beta_k + P}^3} > 0, \quad P \in \mathbb{R}_+. \] (72)

From this result, the objective in (68) is convex and even jointly convex, with respect to each \( \varrho_k \). Since the objective is convex and strictly decreasing with respect to each \( \varrho_k \), a stationary point on the range \( 0 < \varrho_k < P_{\text{max}} \) cannot exist. If there were a stationary point defined by \( (\varrho_1, \varrho_2, \varrho_3, \ldots, \varrho_K) \), then the addition of an \( \varepsilon > 0 \) to the slack variable \( \varrho_k \), which has the smallest slope among all considered slack variables, and subtraction of the same amount \( \varepsilon \) from the slack variable \( \varrho_k \), which has the greatest slope among all considered slack variables, would lead to a greater value of the objective, because of its monotonicity and convexity. However, this would contradict the existence of a stationary point on the range \( 0 < \varrho_k < P_{\text{max}} \). Hence, the optimization of the maximization problem yields a unique optimal value for each slack variable on the boundary of its feasible set. Furthermore, the optimal solution for the slack variable \( \xi \) is zero, i.e., \( \xi^* = 0 \), due to increasing property of the objective (68) with respect to the number of SNs. This means that the first \( K - 1 \) SNs operate on \( P_{\text{max}} \), the \( K \)th SN operates on the remaining power \( P_{\text{remain}} := P_{\text{tot}} - (K - 1)P_{\text{max}} \) with \( 0 < P_{\text{remain}} \leq P_{\text{max}} \), while all other SNs stay idle. Consequently, the optimal power allocation method is simply described as follows.

First, all SNs are re-indexed to satisfy the inequality chain

\[ c_k(P_{\text{max}}) \leq c_{k+1}(P_{\text{max}}), \quad k \in \mathbb{F}_{K-1}. \] (73)

In turn, the first \( K - 1 \) SNs are kept fix while the remaining SNs are re-indexed again to satisfy the inequality chain

\[ c_k(P_{\text{remain}}) \leq c_{k+1}(P_{\text{remain}}), \quad k \in \mathbb{F}_{K-1} \setminus \mathbb{F}_{K-1}. \] (74)

Then, we can conclude

\[ (\varrho_1^*, \ldots, \varrho_{K-1}^*, \varrho_K^*, \varrho_{K+1}^*, \ldots, \varrho_K^*) = (0, \ldots, 0, P_{\text{max}} - P_{\text{remain}}, P_{\text{max}}, \ldots, P_{\text{max}}). \] (75)

From (24), (29), (30), (38), (67) and (68), we infer

\[ V^* = \frac{1}{\frac{1}{P_{\text{rms}}^2} + \sum_{k=1}^{K-1} \frac{1}{c_k(P_{\text{remain}})^2} - 1}, \] (76)

\[ u_k^* = \sqrt{\frac{\alpha_k\beta_k}{\alpha_k + \beta_k + P_{\text{max}}} + \frac{\alpha_k\beta_kP_{\text{remain}}}{\alpha_k + \beta_k + P_{\text{max}}}} - \frac{\alpha_k\beta_k}{\alpha_k + \beta_k + P_{\text{max}}}, \quad k \in \mathbb{F}_{K-1}, \] (77)

\[ u_K^* = \frac{\sqrt{\alpha_K\beta_K}}{\alpha_K + \beta_K + P_{\text{remain}}}, \] (78)

\[ u_k^* = \frac{\sqrt{u_k^* c_k^*}}{\sqrt{\alpha_k + \beta_k + P_{\text{max}}}} + \beta_k, \quad k \in \mathbb{F}_K, \] (79)

\[ W_k^* = \frac{\alpha_k\beta_k u_k^*}{\alpha_k\beta_k u_k^* - (u_k^*)^2}, \quad k \in \mathbb{F}_K, \] (80)

\[ |v_k^*| = \frac{r_{\text{max}}|\beta_k|}{\sqrt{u_k^* c_k^*}} \sqrt{\frac{\alpha_k\beta_k}{\alpha_k + \beta_k + P_{\text{remain}}} - (u_k^*)^2}, \quad k \in \mathbb{F}_K \setminus \mathbb{F}_{K-1}. \] (81)

and

\[ |v_k^*| = W_k^* = u_k^* = 0, \quad k \in \mathbb{F}_K \setminus \mathbb{F}_{K-1}. \] (82)

The number \( K \) of active SNs results from the inequality

\[ 0 < P_{\text{remain}} \leq P_{\text{max}}, \]

that must be fulfilled for the last SN, and is given by the smallest integer number for which the inequality

\[ K \geq P_{\text{tot}} \] (83)

holds.

Note that because of the same argumentation as in Subsection III-B and III-C, the global optimality of the obtained results is ensured.

Note that in the considered case, the fusion rule may be more complicated than in (46), since more SNs are active in general. On the other hand, the fusion rule may be less complicated than that from Subsection III-C, because not all SNs are possibly active.

In summary, equations (75)–(83) and (21) are the optimal solution to the power allocation problem subject to both types of constraints. They are hence the main contribution of the present subsection.

F. Discussion of the solutions

In the case \( P_{\text{max}} < P_{\text{tot}} \leq KP_{\text{max}} \) from Subsection III-E the overall system performance is reduced because of two reasons. First, the SNR of each SN is reduced compared to the results from Subsection III-B, due to the output power-range limitation by \( P_{\text{max}} < P_{\text{tot}} \). Second, not all SNs can be active, due to the sum-power limitation by \( P_{\text{tot}} \leq KP_{\text{max}} \), such that the system performance is weaker compared to the results of Subsection III-C. Hence, the value in (76) is in general greater than those in (40) and (56). This behavior is not surprising and the performance reduction was predictable, since we have included more restrictions into the optimization...
problem. To mention is, that for the general case described in Subsection III-E, we are analytically able to find relationships in closed-form between the disturbance-intensity of each SN, the geographical placement of the corresponding SN, and its operating mode, see [21] and [20] for more information.

In practice, the value of each $c_k$ is in general unique such that the inequality chain $c_k < c_{k+1}$ for all $k \in K - 1$ holds. In this case, the optimal value of the objective (76) is decreasing with respect to both $P_{\text{tot}}$ and $P_{\text{max}}$. If $P_{\text{max}}$ is fixed and $P_{\text{tot}}$ varies in the range $P_{\text{max}} \leq P_{\text{tot}} \leq K P_{\text{max}}$, then the optimal value of the objective (76) is decreasing with respect to $P_{\text{tot}}$ because the SNR of the whole sensor network is increasing with $P_{\text{tot}}$. The best situation is achieved only when all SNs are active, i.e., $P_{\text{tot}} = K P_{\text{max}}$. In contrast, if $P_{\text{tot}}$ is fixed and $P_{\text{max}}$ varies in the range $\frac{1}{K} P_{\text{tot}} \leq P_{\text{max}} \leq P_{\text{tot}}$, then the optimal value of the objective (76) is decreasing with respect to $P_{\text{max}}$ because the capability of each SN is increasing with $P_{\text{max}}$. The best situation is achieved only when a single SN is active, i.e., $P_{\text{max}} = P_{\text{tot}}$.

In a practical application, the value of $P_{\text{max}}$ is fixed and $P_{\text{tot}}$ can suitably be adjusted within the extended range $0 < P_{\text{tot}} \leq K P_{\text{max}}$. In order to save energy, the value of $P_{\text{tot}}$ should be as small as possible, which means that a single SN or only a few SNs are active. On the other hand, to accurately estimate additional quantities such as position, velocity, acceleration, angle of movement, and other important properties and parameters of the target object, more than few SNs are needed to be active. Hence, if the number $K$ of active SNs is satisfactory in order to accurately estimate all important parameters of the target, then the best energy-aware value of $P_{\text{tot}}$ is equal to $K P_{\text{max}}$. In turn, the value of $P_{\text{max}}$ should be large enough to achieve a desired classification or detection probability. With this setup, all system parameters $K$, $P_{\text{max}}$ and $P_{\text{tot}}$ are optimally determined for an energy-aware system design.

Note that all solutions from Subsections III-B, III-C and III-E are different to the well-known water-filling solution, see [22]. The difference to the water-filling solution emerges from the fact that the information flow over each effective path, consisting of a single SN, its sensing channel, the modest signal processing of the same SN, and its communication channel followed by the associated weight in the fusion center, is adjustable due to the power optimization. Thus, on the one hand, the diversity of each effective path is not predetermined such that the water-filling solution cannot hold in its general form. On the other hand, the diversity of best effective paths is amplified in comparison to the diversity of poorest effective paths because of the optimal solution to the power allocation.

**G. Numerical verification**

The main difficulty of dealing with optimization problems like (13), (47) and (60) is the absence of a specific mathematical structure, e.g., monotonicity, convexity and higher order properties. In order to prove or show global optimality, a considerable effort is thus needed for all aforementioned optimization problems, see [23]. In this subsection we present an equivalent numerical method to verify our previous analytical solutions in a simple way. In particular, the optimization problem considered in (60) is rewritten such that to obtain a sequential convex program (SCP), which in turn can easily be solved by standard numerical tools like MATLAB® [24] with the aid of CVX [25]. Our approach is based on the substitution of all variables by

$$ u_k = e^{u'_k}, \quad |v_k| = e^{v'_k}, \quad |w_k| = e^{w'_k}, $$

(84)

where all new variables are real valued. Then an equivalent optimization problem of (61) is given by

**minimize**

$$ \sum_{k \in K} e^{u'_k} \left( e^{2 u'_k} |h_k|^2 M_0 + N_0 \right), $$

s.t.

$$ \sum_{k \in K} e^{u'_k} + e^{v'_k} + e^{w'_k} |g_k h_k| = 1, $$

$$ \sum_{k \in K} \left( 1 + r_{\text{rms}}^2 |g_k|^2 e^{2 u'_k} \right) e^{2 w'_k} + M_0 e^{2 u'_k} \leq P_{\text{tot}}, $$

$$ \left( 1 + r_{\text{rms}}^2 |g_k|^2 e^{2 u'_k} \right) e^{2 w'_k} + M_0 e^{2 u'_k} \leq P_{\text{max}} \forall k, $$

(85)

where the equality constraint is not an affine function. To convexify the above problem, we linearize the equality constraint and obtain the SCP

**minimize**

$$ \sum_{k,n \in K} e^{2 u'_{k,n}} \left( e^{2 u'_{k,n}} |h_{k,n}|^2 M_0 + N_0 \right), $$

s.t.

$$ \sum_{k \in K} e^{\delta_{k,n} - 1} \left( 1 + \delta_{k,n} - \delta_{k,n-1} \right) |g_k h_k| = 1, $$

$$ \sum_{k \in K} \left( 1 + r_{\text{rms}}^2 |g_k|^2 e^{2 u'_{k,n}} \right) e^{2 w'_{k,n}} + M_0 e^{2 u'_{k,n}} \leq P_{\text{tot}}, $$

$$ \left( 1 + r_{\text{rms}}^2 |g_k|^2 e^{2 u'_{k,n}} \right) e^{2 w'_{k,n}} + M_0 e^{2 u'_{k,n}} \leq P_{\text{max}} \forall k, $$

(86)

where the auxiliary variable $\delta_{k,n} := u'_{k,n} + v'_{k,n} + w'_{k,n}$ is used. The counting index $n \in \mathbb{N}$ represents thereby the $n$th solution of the SCP (86) with arbitrary feasible initial-values $u'_{k,0}, v'_{k,0}$ and $w'_{k,0}$. The value of the objective after the $n$th iteration is denoted by $V_n$. For sufficiently large $n$ and an accurate choice of initial-values, we expect that the solution of (86) converges to the solutions derived in Subsection III-E which means $V_n \rightarrow V^*$. In the next section we will discuss these solutions.

**IV. Sensitivity Analysis and Numerical Results**

In this section, we first compare the analytical solution of the objective (76) with the numerical computation of the equivalent SCP (86). Afterwards, we simulatively investigate the behavior of the optimal value in (76) with respect to $\sigma^2_g := E[|g_k|^2]$, $\sigma^2_h := E[|h_k|^2]$, $M_0$ and $N_0$. Subsequently, we analyze the sensitivity of a sensor network, which is indeed designed by the optimal power allocation strategy from Subsection III-E, but with an imperfect knowledge of the actual channel-state. In particular, we investigate different independent cases, where both estimates $g_k := g_k + \Delta g_k$ and $h_k := h_k + \Delta h_k$ are used instead of $g_k$ and $h_k$, respectively, in order to re-design the sensor network. We compare then the optimal value in (76) of the sensor network with optimal known parameters to the conditional mean square error (MSE)

$$ \hat{V} := E[|\bar{r} - r|^2 | \Delta g_k, \Delta h_k] $$

(87)
of the same re-designed sensor network with imperfect information. In general, the estimate \( \hat{r} \) in (87) is biased compared to the case with perfect information, i.e., \( \mathbb{E}[\hat{r} - r_1] \neq 0 \). In addition, the selection of most reliable SNs is no longer ensured. Hence, the value in (87) is mostly greater than that of (76), see also the definition (12). Furthermore, due to inaccurate knowledge of sensing and communication channels, the given power constraints may be violated in the erroneous design. All these effects are equivalently relevant for a discussion and comparison. Since an analytical comparison seems to be out of reach, we set out to use numerical methods to obtain the sensitivity analysis and visualize corresponding simulation results.

In order to fairly compare all results, we simulate a reference curve for each figure. All reference curves are based on parameters under consideration with default values given in Table II. Unless otherwise stated, we usually create a new curve only by changing the value of a single parameter. The specific new value of that parameter is noted in the legend of the corresponding figure.

### Table II
**Default Values Used for Each Reference Curve.**

<table>
<thead>
<tr>
<th>( K )</th>
<th>( r_{\text{rms}}^2 )</th>
<th>( \sigma_g^2 )</th>
<th>( \sigma_h^2 )</th>
<th>( \sigma_{\Delta \rho}^2 )</th>
<th>( \sigma_{\Delta h}^2 )</th>
<th>( M_0 )</th>
<th>( N_0 )</th>
<th>( P_{\text{max}} )</th>
<th>( P_{\text{tot}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>10</td>
</tr>
</tbody>
</table>

### A. Verification of analytical and numerical solutions

In Figure 3, both analytical and numerical results for the optimal objective respectively obtained by (76) and (86) are presented. The SCP is calculated with three criteria of termination. The first one is a minimum number of iterations \( n \geq 8 \). The second criterion is a feasibility check, fulfilled by

\[
|1 - \sum_{k \in \mathcal{M}_K} e^{|\delta_{k,m} - \delta_{k,n}|} |g_k h_k| | \leq 10^{-4}.
\]

The last one is a check of convergency and is determined by the relative value condition \( |V_n - V_{n-5}| \leq V_n \cdot 10^{-5} \). All above criteria are based on experience in this field. Furthermore, each SCP point is calculated with three randomly and independently generated initial-values, where at the end of each three runs the best achieved result is depicted. Since the precision of the numerical solutions are very high, due to the strict termination criteria from above, all SCP points fall onto the analytical curves. This coincidence reinforces the statement for global optimality of the analytical results on the one hand, and the convergence of the numerical method on the other hand.

Since the solution of the SCP in (86) is highly computationally intensive and its results do not show any additional information, we omit this computation and its corresponding results in the remaining part of this paper.

### B. Behavior of \( V^* \)

By considering (38) we see that the disturbance-intensity \( \sigma_k(P) \) is symmetric in \( \alpha_k \) and \( \beta_k \) and hence the optimum value \( V^* \) in (76) is also symmetric in each \( \alpha_k \) and \( \beta_k \). Thus, changing the roles of each pair \( g_k \) and \( M_0 \) with each pair \( h_k \) and \( N_0 \) results in the same value for \( V^* \) while keeping \( r_{\text{rms}} = 1 \) constant. Because of this fact it is enough to investigate the behavior of \( V^* \) only with respect to \( \sigma_g^2 \) and \( M_0 \) while \( \sigma_h^2, N_0 \) and \( r_{\text{rms}} \) are kept constant. All random processes \( g_k, h_k, \Delta g_k, \Delta h_k, m_k \) and \( n_k \) are randomly generated with zero mean Gaussian distributions for each simulation step. The random process \( r_1 \) is randomly generated with a uniform distribution on \( \{-1, 1\} \) in each simulation step. All other parameters are kept constant. We apply a Monte-Carlo simulation, where the number of parameter realizations for each simulation point is always 100000. The same simulation setup is also applied for all results in the next subsection.

In Figure 4, the decreasing property of \( V^* \) with respect to the variance \( \sigma_g^2 \) of all sensing channels is shown. The reason
behind the decreasing property is that the whole network observes the target object more reliable in case where the variance of sensing channels is higher. Furthermore, it can be seen that increasing \( N_0 \) has an equivalent effect on the objective as decreasing \( \sigma^2_h \) and vice versa. It is also interesting to note that the objective shows highest sensitivity to the variation of \( M_0 \) when the sensing channel is rather weak (small \( \sigma^2_\hat{g} \)). The observed sensitivity is illustratively reduced for higher variances of the sensing channel. Furthermore, the objective attains a more or less constant value for very high variances of the sensing channel. The reason is that the resulted objective is dominated by the quality of the communication channel when the sensing channel gets stronger.

In Figure 5 it is shown that in contrast to the curves in Figure 4, the property of \( V^* \) is increasing with respect to the noise power \( M_0 \). For small values of \( M_0 \) all curves have a square root property while for large values of \( M_0 \) all curves behave linear. The deviation of all curves is greater for large values of \( M_0 \) than for small values. Furthermore, the value of \( M_0 \) has more impact on the deviation of \( V^* \) caused by \( \sigma^2_g \) than by other parameters, as mentioned before. As already described for Figure 4, increasing \( N_0 \) is equivalent to decreasing \( \sigma^2_h \) and vice versa.

**C. Sensitivity of \( \hat{V} \)**

A sensitivity analysis of \( \hat{V} \) is very important in order to justify assumptions concerning the channel-state knowledge. In Figure 6, we consider the case where the error variance \( \sigma^2_{\Delta g} := \mathbb{E}[|\Delta g_k|^2] \) of estimated sensing channels is greater than or equal to zero. In the case where \( \sigma^2_{\Delta g} \) is equal to zero, the identity \( \hat{V} = V^* \) holds. Otherwise, \( V \) is always greater than \( V^* \), i.e., \( \hat{V}(\sigma^2_{\Delta g}) \geq V(0) \). All curves pass through three different regions whereas only the first and the second region are visible in Figure 6. In the first region, i.e., for low values of \( \sigma^2_{\Delta g} \), all curves are slowly increasing. The selection of most reliable SNs is still mostly ensured in the first region while the optimal power allocation is no longer insured. In the second region, i.e., for mid-range values of \( \sigma^2_{\Delta g} \), all curves are rapidly increasing since the correct sensor selection gets out of control. In the last region, i.e., for high values of \( \sigma^2_{\Delta g} \), all curves are linearly increasing. In this region the optimal sensor selection almost always fails. This means that SNs are randomly selected and the allocated power is also random. The increasing property (not constant) of all curves in the third region is comparable with the increasing property in the first region, because the roles of \( g_k \) and \( \Delta g_k \) are exchanged. The system is then designed by \( \Delta g_k \) instead of \( g_k \) and thus \( g_k \) itself acts as an estimation error of \( \Delta g_k \). In summary, the best operation region of the proposed system is the first region in which \( \sigma^2_{\Delta g} \ll \sigma^2_g \) holds, while a system operation in the third region, for which \( \sigma^2_{\Delta g} \gg \sigma^2_g \) holds, should be avoided.

As can also be seen, the deterioration of the performance is not only amplified by high noise powers \( M_0 \) and \( N_0 \), but also by low channel variances \( \sigma^2_g \) and \( \sigma^2_h \). The form of the curves is mainly dominated and prescribed by the channel variance \( \sigma^2_g \). All curves in the middle of the figure run almost parallel because the variance \( \sigma^2_g \) of all those curves is the same.

**V. Conclusion**

The main contribution of the present work is to present an optimal solution to the power allocation problem in distributed active multiple-radar systems subject to different power constraints. We have introduced a system model, a linear fusion rule and a simple objective function, which enable us to solve the power allocation problem analytically. Three different cases of power constraints have been investigated. For a limitation of transmission power per sensor node and a sum-power limitation as well as their combination, we have analytically obtained optimal solutions in closed-form. Furthermore, all proposed solutions are valid for AWGN channels as well as

![Fig. 5. Behavior of \( V^* \) with respect to \( M_0 \). All curves show an increasing property in \( M_0 \). The reference curve has the default parameters \( \sigma^2_g = 2, \sigma^2_h = 2 \) and \( N_0 = 2 \).](image1)

![Fig. 6. Behavior of \( \hat{V} \) with respect to \( \sigma^2_{\Delta g} \). All curves show an increasing property in \( \sigma^2_{\Delta g} \). The reference curve has the default parameters \( \sigma^2_g = 2, \sigma^2_h = 2, M_0 = 2 \) and \( N_0 = 2 \).](image2)
for frequency-flat slow-fading channels, provided that channel state information is available at each receiver. In addition, we have discussed the sensitivity of the derived optimal power allocation with respect to perfect and imperfect parameter knowledge. In this way, we have shown that the optimal power allocation is robust against small inaccuracies in the channel state information.

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