Capacity Regions and Optimal Power Allocation for CDMA Cellular Radio

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***Abstract*—The capacity region of code-division multiple access (CDMA) is determined by the set of transmission rates combined with quality-of-service (QoS) requirements which allow for a feasible power allocation scheme for n mobiles in a cellular network. The geometrical and topological properties of the capacity region are investigated in the present paper for the case of unlimited and limited power, respectively. As a central result, we show that the capacity region is convex by breaking the complicated topological structure into characteristic properties of its boundary and interior points, each of interest in itself. Based on these results, we furthermore investigate optimal power assignment schemes in the case that the demand of a community of users is infeasible. Weighted minimax and Bayes solutions are explicitly determined as appropriate means to share the capacity of a cellular network in a reasonable and fair way.**

***Index Terms*—Bayes strategy, capacity region, cellular networks, code-division multiple access (CDMA), convexity, minimax strategy, optimal power control, Perron–Frobenius theory.**

I. Introduction

POWER control is an essential building block of code-division multiple-access (CDMA) cellular radio for achieving the high potential capacity. Each mobile calibrates its transmission power according to the desired data rate, quality of transmission, and path loss to the linking base station such that a certain minimal bit energy-to-noise ratio is achieved. If the chip rate of the system is fixed, the data rate and quality of transmission jointly determine the effective spreading gain of each user. For a fixed connectivity of n mobiles to linking base stations, and corresponding fixed path losses, we define the capacity region as the set of effective spreading gain vectors, which allow for a feasible power assignment such that everybody receives the data rate and quality of transmission as required.

In this work, we thoroughly investigate the geometry and topological properties of the capacity region thus defined for the case of unlimited and limited transmit power, respectively. For both cases, the capacity region is shown to be a convex set. A number of related results has been published in recent years, each with different intentions, and using different methods of proof. Sung [1] was the first to show log-convexity of the feasible signal-to-interference ratio (SIR) region neglecting power constraints. In [2], a generalization to log-convex transformations of certain quality-of-service (QoS) requirements is given. The work [3] by the same authors includes power constraints and also investigates the convergence behavior at boundary points. Independently, in [4], the convexity of the capacity region is derived. This paper also gives an interesting duality between up- and downlink, deals with computational aspects, and includes a stochastic model for power control when path gain is assumed to be a random variable.

To achieve the results in the present paper, a number of preparatory results, each of interest in itself, is needed. We deal with supporting hyperplanes and exposed points, and moreover, interior and boundary points of the capacity region. The geometrical characterization and convexity pave the way for determining optimal resource sharing strategies in the unlimited and limited case. For any infeasible demand profile, we seek a closest feasible power assignment in the sense of minimizing the maximum deviation. Since the capacity region for unlimited power is an open set, the above results can be achieved only up to some allowable deviation. We therefore introduce the concept of ε-minimax allocations and then provide a complete solution to the corresponding ε-minimax allocation problem. Besides these minimax strategies we also determine ε-Bayes power allocations, which means to minimize a weighted mean of effective spreading gains over the capacity region. Both strategies describe best effort power balancing schemes for infeasible user demands, which are particularly suitable for data applications where the target QoS is negotiable.

Our optimality criteria are based on QoS requirements in terms of the effective spreading gain. A direct approach by using the signal-to-interference plus noise ratio (SINR) directly is also conceivable. In this case, however, convexity is lost, see, e.g., [3], and the present general methods for power balancing would be no more applicable.

Capacity and optimal power control are closely related such that the following works are of importance to the present paper. Reference [5] was one of the first to address the power control problem analytically as a minmax interference balancing problem. The method of considering the power control and assignment problem for CDMA by linear equations with positive solutions has been used in [6]. In this paper, the existence of a feasible power control vector is clarified by use of Perron–Frobenius' theory and, furthermore, a provably convergent algorithm is presented for assigning mobiles to base stations.

In an excellent survey article, the authors [7] consider power control as a flexible mechanism to ensure QoS demands of individual users. Mainly two questions are studied, namely, optimal

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power control and characterizing the resulting network capacity under different receiver designs.

Power control algorithms are extended to imprecise measurements via stochastic modeling in [8]. Two classes of distributed power control algorithms are introduced and their mean square convergence is shown.

We partly use methods akin to [9]. The carrier-to-interference ratio is of equivalent importance in the beamforming concept. The crosstalk terms between stations depend on the channel and beamforming vectors, and directly relate to the path loss coefficients in this paper.

This paper is organized as follows. Section II contains some prerequisites and basic notation. In Section III, we study the boundary of capacity regions without power constraints. This provides a fairly complete description of the geometry of these regions, which we use in Section IV to derive explicit expressions of ε-optimal power allocations in the unconstrained case. In Section V, we establish a close relationship between capacity regions with power constraints and certain capacity regions without constraints but with suitably modified path losses. This key result opens the door to understanding the rather more complex constraint case. We show in particular that the capacity region remains convex in the presence of power constraints. The results thus obtained are used in Section VI to construct a minimax power allocation, that is, a best compromise between contrary interests.

II. Notation and Prerequisites

We start by introducing the basic notation, for an overview see Table I. Assume a CDMA system with chip rate ω, e.g., $\omega = 3.84$ MChip/s for the universal mobile telecommunications system (UMTS). Each user $i \in \{1,\dots,n\}$ has a certain data rate R_i to transmit and requires an individual minimum bit-error rate. Let $\tilde{s}_i = \omega/R_i$ denote the spreading gain. Since the bit-error rate is a function of the bit energy-to-noise ratio, E_b/N_0, individual quality demands can be described by lower bounds e_i as follows:

$$\left(\frac{E_b}{N_0}\right)_i = \tilde{s}_i \left(\frac{C}{I}\right)_i \geq e_i \tag{1}$$

where C/I denotes the carrier-to-interference ratio at the mobile's connecting base station.

In the following, we assume a fixed allocation of mobiles to base stations, expressed by an assignment function

$$c : \{1,\dots,n\} \to \{1,\dots,K\} : i \mapsto k_i$$

such that k_i denotes i's connecting base station. The set of mobiles allocated to base station k is denoted by $\mathcal{C}(k) = \{i \mid k_i = k\}$, $k = 1,\dots,K$. $\mathcal{C}(k)$ is simply a partition of the set $\{1,\dots,n\}$.

In the uplink, let p_i denote the transmit power of mobile i, and $a_{ik} \in [0,1]$ the path loss from mobile i to base station k. We assume that $a_{ik} > 0$ for all $i \in \mathcal{C}(k)$, which is obvious to avoid meaningless assignments. Using the effective spreading gain $s_i = \tilde{s}_i/e_i$, (1) reads as

$$s_i \left(\frac{C}{I}\right)_i = s_i \, \frac{p_i a_{ik_i}}{\sum_{j\neq i} p_j a_{jk_i} + \tilde{\tau}_{k_i}} \geq 1, \qquad i = 1,\dots,n. \tag{2}$$

TABLE I
Basic Notations

$k \in \{1,\dots,K\}$	Labeling of base stations
$i \in \{1,\dots,n\}$	Labeling of mobile stations
k_i	Base station allocated to mobile i
$\mathcal{C}(k) \subseteq \{1,\dots,n\}$	Set of mobiles allocated to base station k
ω	Chip rate of the CDMA system
R_i	Data rate of user i
$\tilde{s}_i = \omega/R_i$	Spreading gain of user i
e_i	Lowest acceptable transmission quality of user i
$s_i = \tilde{s}_i/e_i$	Effective spreading gain of user i
p_i	Transmit power of user i
$\hat{p}_i$	Maximum transmit power of user i
$a_{ik} \in [0,1]$	Path loss from mobile i to base station k
$\tilde{\tau}_k$	Background and thermal noise at base station k
$\tau_i = \tilde{\tau}_{k_i}/a_{ik_i}$	Relative background noise for mobile i
$\boldsymbol{\pi}(\boldsymbol{s})$	Power allocation corresponding to $\boldsymbol{s}$
$\boldsymbol{\sigma}(\boldsymbol{p})$	Vector of spreading gains corresponding to $\boldsymbol{p}$
$\boldsymbol{t}$	Some infeasible demand profile
$\hat{\boldsymbol{p}}$	Vector of transmit power bounds
$\boldsymbol{w}$	Weight vector
$\rho(\boldsymbol{A})$	Spectral radius of matrix $\boldsymbol{A}$

The numerator $p_i a_{ik_i}$ represents the received power of mobile i at the connecting base station k_i, $\sum_{j\neq i} p_j a_{jk_i}$ collects the received interference from all other mobiles, and $\tilde{\tau}_{k_i} > 0$ denotes the general background and thermal receiver noise at base station k_i. s_i, $i = 1,\dots,n$, combines the user demands, namely, the data rate R_i and the quality of transmission e_i into a single quantity.

Only the minmum transmit power is of interest such that (2) is satisfied. Since the numerator of (2) is increasing in p_i and the denominator is increasing in p_j, $j \neq i$, it is clear that the minimum is attained at the boundary such that a solution $\boldsymbol{p} = (p_i)_{1\leq i\leq n}$ of the system

$$s_i \, \frac{p_i a_{ik_i}}{\sum_{j\neq i} p_j a_{jk_i} + \tilde{\tau}_{k_i}} = 1, \qquad i = 1,\dots,n$$

is needed. The above equations are easily converted into the following system:

$$p_i - \sum_{j\neq i} \frac{a_{jk_i}}{s_i a_{ik_i}} \, p_j = \frac{\tilde{\tau}_{k_i}}{s_i a_{ik_i}}, \qquad i = 1,\dots,n. \tag{3}$$

Collecting the user demands s_i into a diagonal matrix

$$\boldsymbol{S} = \mathrm{diag}\big(s_1,\dots,s_n\big)$$

using the notation

$$\boldsymbol{B} = \big(b_{ij}\big)_{i,j=1,\dots,n}, \qquad \text{with } b_{ij} = \begin{cases} 0, & \text{if } i = j \\ \frac{a_{jk_i}}{a_{ik_i}}, & \text{if } i \neq j \end{cases}$$

and

$$\boldsymbol{\tau} = (\tau_1,\dots,\tau_n)' = \left(\frac{\tilde{\tau}_{k_1}}{a_{1k_1}},\dots,\frac{\tilde{\tau}_{k_n}}{a_{nk_n}}\right)'$$

the system of linear (3) can be written as

$$\big(\boldsymbol{I} - \boldsymbol{S}^{-1}\boldsymbol{B}\big)\,\boldsymbol{p} = \boldsymbol{S}^{-1}\,\boldsymbol{\tau}. \tag{4}$$

In the following, the notation "$>$" and "$\geq$" for vectors and matrices means that the corresponding relations hold elementwise. Furthermore, let $\rho(\boldsymbol{A})$ denote the spectral radius of a square matrix $\boldsymbol{A}$, i.e., the maximum of the absolute values of all complex eigenvalues of $\boldsymbol{A}$. An immediate consequence of Perron–Frobenius' theory (see [10, p. 30]) is the following result.

Proposition 1: Suppose that $\boldsymbol{B}$ is irreducible. If $\rho\left(\boldsymbol{S}^{-1}\boldsymbol{B}\right) < 1$, then (4) has a unique solution $\boldsymbol{p} \in \mathbb{R}^n$ and $\boldsymbol{p} > 0$. If $\rho\left(\boldsymbol{S}^{-1}\boldsymbol{B}\right) \geq 1$, then (4) does not have any nonnegative solution $\boldsymbol{p}$.

This gives rise to define the set of feasible user demands as the set of vectors $\boldsymbol{s} = (s_1, \ldots, s_n)' \in \mathbb{R}_+^n$ that allow for an admissible power allocation, and hence can be served by the network.

$$\mathcal{S}_{\boldsymbol{B}} = \left\{\boldsymbol{s} > \boldsymbol{0} \mid \rho\big([\mathrm{diag}(\boldsymbol{s})]^{-1}\boldsymbol{B}\big) < 1\right\} \tag{5}$$

is called the set of feasible uplink demands, or the uplink capacity region of the network. Subscript $\boldsymbol{B}$ refers to the matrix $\boldsymbol{B}$. In the sequel, the involved matrix will vary. Throughout the paper, it is assumed that $\boldsymbol{B}$ is nonnegative and irreducible.

In view of Proposition 1, there is a one-to-one correspondence between the feasible demands and the power allocations. Explicitly, for every $\boldsymbol{s} \in \mathcal{S}_{\boldsymbol{B}}$ there is a unique power allocation vector $\boldsymbol{\pi}(\boldsymbol{s}) = (\pi_1(\boldsymbol{s}), \ldots, \pi_n(\boldsymbol{s}))' > \boldsymbol{0}$ such that (4) holds with $\boldsymbol{S} = \mathrm{diag}(\boldsymbol{s})$ and $\boldsymbol{p} = \boldsymbol{\pi}(\boldsymbol{s})$. Conversely, multiplying (4) by $\boldsymbol{S}$, one obtains the equation $(\boldsymbol{S} - \boldsymbol{B})\boldsymbol{p} = \boldsymbol{\tau}$, and it follows that for every $\boldsymbol{p} > 0$ there is a unique $\boldsymbol{\sigma}(\boldsymbol{p}) = (\sigma_1(\boldsymbol{p}), \ldots, \sigma_n(\boldsymbol{p}))' \in \mathcal{S}_{\boldsymbol{B}}$ such that (4) holds with $\boldsymbol{S} = \mathrm{diag}(\boldsymbol{\sigma}(\boldsymbol{p}))$. The functions $\boldsymbol{\sigma}$ and $\boldsymbol{\pi}$ satisfy

$$[\mathrm{diag}(\boldsymbol{\sigma}(\boldsymbol{p})) - \boldsymbol{B}]\,\boldsymbol{p} = \boldsymbol{\tau}, \quad \boldsymbol{\pi}(\boldsymbol{\sigma}(\boldsymbol{p})) = \boldsymbol{p} \quad \text{for all } \boldsymbol{p} > \boldsymbol{0} \tag{6}$$

$$[\mathrm{diag}(\boldsymbol{s}) - \boldsymbol{B}]\,\boldsymbol{\pi}(\boldsymbol{s}) = \boldsymbol{\tau}, \quad \boldsymbol{\sigma}(\boldsymbol{\pi}(\boldsymbol{s})) = \boldsymbol{s} \quad \text{for all } \boldsymbol{s} \in \mathcal{S}_{\boldsymbol{B}}. \tag{7}$$

This yields in particular the following parametric representation of the capacity region:

$$\mathcal{S}_{\boldsymbol{B}} = \{\boldsymbol{\sigma}(\boldsymbol{p}) \mid \boldsymbol{p} > 0\}, \ \text{ where } \sigma_i(\boldsymbol{p}) = \frac{1}{p_i}\Big(\tau_i + \sum_{j \neq i} b_{ij}\, p_j\Big). \tag{8}$$

From (5) it is obvious that $\mathcal{S}_{\boldsymbol{B}}$ is independent of the relative background noise $\boldsymbol{\tau}$. This is due to the fact that there is no upper bound on the transmit power of mobiles. In Section V, we will include restrictions on the maximum transmission power and investigate the structure of the capacity region $\mathcal{S}_B$ under this additional constraint. It turns out that there is an intimate connection between any given capacity region with power restrictions and a suitably selected capacity region with unrestricted power.

III. Capacity Regions Without Power Constraints

In this section, we study the geometry of the capacity region $\mathcal{S}_{\boldsymbol{B}}$. The results will be applied to construct optimal power allocations and will play a fundamental role in studying the capacity region under power constraints.

A result of central importance is the following, its proof is given in [4].

Proposition 2: $\mathcal{S}_{\boldsymbol{B}}$ is an open convex subset of $\mathbb{R}_+^n$.

The boundary points of $\mathcal{S}_{\boldsymbol{B}}$ are of particular interest since they represent boundary states of the system where no additional capacity can be provided. Close to such points, contradicting interests of users must be somehow balanced taking account of individual demands and utilities.

Proposition 3:

a) The set of boundary points $\partial\mathcal{S}_{\boldsymbol{B}}$ of $\mathcal{S}_{\boldsymbol{B}}$ is given by

$$\partial\mathcal{S}_{\boldsymbol{B}} = \left\{\boldsymbol{s} > \boldsymbol{0} \mid \rho\big([\mathrm{diag}(\boldsymbol{s})]^{-1}\boldsymbol{B}\big) = 1\right\} \tag{9}$$

$$= \Big\{\boldsymbol{s} > \boldsymbol{0} \mid s_i = \frac{1}{p_i}\sum_{j \neq i} b_{ij}\, p_j, \ \ p_1, \ldots, p_n > 0\Big\}. \tag{10}$$

b) If $\{\boldsymbol{s}(k)\}_{k=1}^{\infty}$ is a sequence of points in $\mathcal{S}_{\boldsymbol{B}}$ that converges to a boundary point $\boldsymbol{s}^*$, then the corresponding power allocations behave as follows. For all $i = 1, \ldots, n$

$$\lim_{k\to\infty} \pi_i(\boldsymbol{s}(k)) = \infty \text{ and } \lim_{k\to\infty} \frac{\pi_i(\boldsymbol{s}(k))}{\sum_{j=1}^n \pi_j(\boldsymbol{s}(k))} = p_i^* \tag{11}$$

where $(p_1^*, \ldots, p_n^*)' = \boldsymbol{p}^*$ is the Perron vector of $[\mathrm{diag}(\boldsymbol{s}^*)]^{-1}\boldsymbol{B}$, the Perron root being 1; that is,

$$[\mathrm{diag}(\boldsymbol{s}^*)]^{-1}\boldsymbol{B}\boldsymbol{p}^* = \boldsymbol{p}^* \quad \text{and} \quad \sum_{j=1}^n p_j^* = 1.$$

Proof: We first prove b). Write

$$\boldsymbol{S}(k) = \mathrm{diag}(\boldsymbol{s}(k)) \quad \text{and} \quad \boldsymbol{S}^* = \mathrm{diag}(\boldsymbol{s}^*).$$

Thus, $\boldsymbol{S}(k) \to \boldsymbol{S}^*$ and, by (7)

$$[\boldsymbol{S}(k) - \boldsymbol{B}]\boldsymbol{\pi}(\boldsymbol{s}(k)) = \boldsymbol{\tau}, \qquad k = 1, 2, \ldots. \tag{12}$$

If $\sum_{j=1}^n \pi_j(\boldsymbol{s}(k)) \not\to \infty$, there would exist a subsequence $k_1 < k_2 < \cdots$ and some $\boldsymbol{q} \geq \boldsymbol{0}$ such that $\lim_{l\to\infty} \boldsymbol{\pi}(\boldsymbol{s}(k_l)) = \boldsymbol{q}$. But then, from (12), $[\boldsymbol{S}^* - \boldsymbol{B}]\boldsymbol{q} = \boldsymbol{\tau}$. In particular $\boldsymbol{s}^* > 0$ and it would follow from Proposition 1 that the boundary point $\boldsymbol{s}^*$ belongs to the open set $\mathcal{S}_{\boldsymbol{B}}$, which is impossible. Hence, $\sum_{j=1}^n \pi_j(\boldsymbol{s}(k)) \to \infty$.

Now set

$$\boldsymbol{r}(k) = \left[\sum_{j=1}^n \pi_j(\boldsymbol{s}(k))\right]^{-1} \boldsymbol{\pi}(\boldsymbol{s}(k)).$$

It will be shown that every convergent subsequence of $\{\boldsymbol{r}(k)\}_{k=1}^{\infty}$ necessarily has the same limit, namely, $\boldsymbol{p}^*$. This clearly implies that the whole sequence converges to $\boldsymbol{p}^*$. Thus, suppose that $\boldsymbol{r}(k_l) \to \boldsymbol{r}^*$ for some $\boldsymbol{r}^* \geq \boldsymbol{0}$. Since $\sum_{j=1}^n \pi_j(\boldsymbol{s}(k_l)) \to \infty$, it follows from (12) that $\boldsymbol{S}^*\boldsymbol{r}^* = \boldsymbol{B}\boldsymbol{r}^*$. To conclude from this equation that $\boldsymbol{r}^*$ coincides with the Perron vector $\boldsymbol{p}^*$, and in fact to prove the very existence of that Perron vector, it has to be shown that $\boldsymbol{s}^* > 0$. To this end, consider the index set $\mathcal{I} = \{i \mid r_i^* = 0\}$. Assume that $\mathcal{I} \neq \emptyset$. Since $\sum_{i=1}^n r_i^* = 1$, $\mathcal{I}^C = \{i \mid r_i^* > 0\} \neq \emptyset$. Thus, since $\boldsymbol{B}$ is irreducible, there exist $i_0 \in \mathcal{I}$ and $j_0 \in \mathcal{I}^C$ such that $b_{i_0 j_0} > 0$. Consequently

$$0 = s_{i_0}^* r_{i_0}^* = \sum_{j=1}^n b_{i_0 j} r_j^* \geq b_{i_0 j_0} r_{j_0}^* > 0$$

which is a contradiction. Therefore, $\mathcal{I} = \emptyset$, so that $\boldsymbol{B}\boldsymbol{r}^* > 0$. It now follows that indeed $\boldsymbol{s}^* > 0$ and $\boldsymbol{S}^{*-1}\boldsymbol{B}\boldsymbol{r}^* = \boldsymbol{r}^*$. That is, $\boldsymbol{r}^*$ is a positive eigenvector of the nonnegative matrix $\boldsymbol{S}^{*-1}\boldsymbol{B}$ corresponding to the eigenvalue 1. This implies that the Perron root $\rho(\boldsymbol{S}^{*-1}\boldsymbol{B}) = 1$, see [11, Corollary 8.1.30, p. 493]. As $\boldsymbol{S}^{*-1}\boldsymbol{B}$ is irreducible, the Perron root is a simple eigenvalue, and so

$\boldsymbol{r}^* = \boldsymbol{p}^*$. This proves the second limit assertion in (11). The first one follows from the second since $\boldsymbol{p}^* > 0$ and $\sum_{j=1}^n \pi_j(\boldsymbol{s}(k)) \to \infty$.

To prove a), first note that if $\boldsymbol{s} \in \partial\mathcal{S}_{\boldsymbol{B}}$, then, as we have just shown, $\boldsymbol{s} > 0$ and $\text{diag}(\boldsymbol{s})\boldsymbol{p} = \boldsymbol{B}\boldsymbol{p}$ for some $\boldsymbol{p} > 0$. That is, $\boldsymbol{s}$ is an element of the set (10). Next, if $\boldsymbol{s}$ belongs to this set, then $[\text{diag}(\boldsymbol{s})]^{-1}\boldsymbol{B}\boldsymbol{p} = \boldsymbol{p}$ for some $\boldsymbol{p} > 0$, and so, again by [11, Corollary 8.1.30], $\rho([\text{diag}(\boldsymbol{s})]^{-1}\boldsymbol{B}) = 1$. That is, $\boldsymbol{s}$ is an element of the right-hand side of (9). Finally, if $\rho([\text{diag}(\boldsymbol{s})]^{-1}\boldsymbol{B}) = 1$, then $\boldsymbol{s} \notin \mathcal{S}_{\boldsymbol{B}}$, but for every $\alpha > 1$

$$\rho([\text{diag}(\alpha\boldsymbol{s})]^{-1}\boldsymbol{B}) = \alpha^{-1} < 1$$

so that $\alpha\boldsymbol{s} \in \mathcal{S}_{\boldsymbol{B}}$. That is, $\boldsymbol{s} \in \partial\mathcal{S}_{\boldsymbol{B}}$. □

The parametric representation (10) establishes a natural one-to-one correspondence between the boundary points of $\mathcal{S}_{\boldsymbol{B}}$ and the rays of the form $\{\lambda\boldsymbol{p}^* \mid \lambda > 0\}$ with $\boldsymbol{p}^* > 0$. To all points $\boldsymbol{p}$ in one ray there corresponds the same boundary point $\boldsymbol{s}$ with $s_i = p_i^{-1}\sum_{j\neq i} b_{ij}p_j$. For every boundary point $\boldsymbol{s}^*$, the corresponding ray is the set of positive multiples of the Perron vector of $[\text{diag}(\boldsymbol{s}^*)]^{-1}\boldsymbol{B}$. Part b) of Proposition 3 says that as $\boldsymbol{s}(k) \to \boldsymbol{s}^*$, the power allocations $\boldsymbol{\pi}(\boldsymbol{s}(k))$ move in the direction of the ray corresponding to the limit point $\boldsymbol{s}^*$ toward a horizon point.

As the set $\mathcal{S}_{\boldsymbol{B}}$ is convex, so is its closure $\overline{\mathcal{S}}_{\boldsymbol{B}} = \mathcal{S}_{\boldsymbol{B}} \cup \partial\mathcal{S}_{\boldsymbol{B}}$. For any $\boldsymbol{s}^* \in \partial\mathcal{S}_{\boldsymbol{B}}$, there exists, therefore, a supporting hyperplane, i.e., there exists some inward normal $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{s}^*) \neq \boldsymbol{0}$, such that $\boldsymbol{y}'(\boldsymbol{s} - \boldsymbol{s}^*) \geq 0$ for all $\boldsymbol{s} \in \overline{\mathcal{S}}_{\boldsymbol{B}}$. Furthermore, it holds that $\boldsymbol{y} \geq 0$. For suppose that $y_i < 0$ for some index i. By choosing $p_i = \varepsilon$, $p_j = 1$, $j \neq i$, all s_j, $j \neq i$ are upper-bounded independent of ε. However, $s_i \to \infty$ as $\varepsilon \to 0$ such that there is some $\hat{\boldsymbol{s}}(\varepsilon)$ with $\boldsymbol{y}'(\hat{\boldsymbol{s}} - \boldsymbol{s}^*) < 0$, a contradiction. The above conclusions resemble a standard argument of statistical decision theory (e.g., [12, p. 87]).

The following proposition provides more detailed information on the supporting hyperplane and an explicit formula for the normal. The result is crucial for analyzing ε-optimal allocations in the uncontrained case, see Theorems 1 and 2. The result will also be used to prove some geometric properties of the capacity region with power constraints, see Propositions 6 and 8.

Proposition 4: Through every boundary point $\boldsymbol{s}^*$ of the closed convex set $\overline{\mathcal{S}}_{\boldsymbol{B}}$, there passes a unique supporting hyperplane to $\overline{\mathcal{S}}_{\boldsymbol{B}}$. The corresponding inward normal is (a positive multiple of) $\boldsymbol{y}^* = (p_1^*q_1^*, \ldots, p_n^*q_n^*)'$, where $(p_1^*, \ldots, p_n^*)' = \boldsymbol{p}^*$ and $(q_1^*, \ldots, q_n^*)' = \boldsymbol{q}^*$ are the Perron vectors of $[\text{diag}(\boldsymbol{s}^*)]^{-1}\boldsymbol{B}$ and $[\text{diag}(\boldsymbol{s}^*)]^{-1}\boldsymbol{B}'$, respectively. In particular, $\boldsymbol{y}^* > 0$. Moreover, every boundary point is an exposed point of $\overline{\mathcal{S}}_{\boldsymbol{B}}$, that is,

$$\boldsymbol{y}^{*\prime}(\boldsymbol{s} - \boldsymbol{s}^*) > 0, \qquad \text{for all } \boldsymbol{s} \in \overline{\mathcal{S}}_{\boldsymbol{B}} \setminus \{\boldsymbol{s}^*\}.$$

Proof: Let $\boldsymbol{y}$ be any inward normal to $\overline{\mathcal{S}}_{\boldsymbol{B}}$ at $\boldsymbol{s}^* \in \partial\mathcal{S}_{\boldsymbol{B}}$. Then, in particular, $\boldsymbol{y}'(\boldsymbol{s} - \boldsymbol{s}^*) \geq 0$ for all $\boldsymbol{s} \in \partial\mathcal{S}_{\boldsymbol{B}}$. In view of Proposition 3 a), this means that

$$h(\boldsymbol{p}) = \sum_{i=1}^n \frac{y_i}{p_i}\sum_{j\neq i} b_{ij}p_j - \sum_{i=1}^n \frac{y_i}{p_i^*}\sum_{j\neq i} b_{ij}p_j^* \geq 0$$

for all $\boldsymbol{p} > \boldsymbol{0}$, and $h(\boldsymbol{p}^*) = 0$. Hence, for $k = 1, \ldots, n$

$$0 = \left.\frac{\partial h(\boldsymbol{p})}{\partial p_k}\right|_{\boldsymbol{p}=\boldsymbol{p}^*} = \sum_{i\neq k} \frac{y_i}{p_i^*}b_{ik} - \frac{y_k}{p_k^{*2}}\sum_{j\neq k} b_{kj}p_j^*$$

$$= \sum_{i\neq k} \frac{y_i}{p_i^*}b_{ik} - \frac{y_k}{p_k^*}s_k^*$$

or, equivalently

$$[\boldsymbol{B}' - \text{diag}(\boldsymbol{s}^*)][\text{diag}(\boldsymbol{p}^*)]^{-1}\boldsymbol{y} = \boldsymbol{0}.$$

That is, $[\text{diag}(\boldsymbol{p}^*)]^{-1}\boldsymbol{y}$ is an eigenvector of the irreducible matrix $[\text{diag}(\boldsymbol{s}^*)]^{-1}\boldsymbol{B}'$ corresponding to the eigenvalue 1. By Proposition 3 a)

$$\rho([\text{diag}(\boldsymbol{s}^*)]^{-1}\boldsymbol{B}') = \rho(\boldsymbol{B}'[\text{diag}(\boldsymbol{s}^*)]^{-1}) = \rho([\text{diag}(\boldsymbol{s}^*)]^{-1}\boldsymbol{B}) = 1$$

and it follows that $[\text{diag}(\boldsymbol{p}^*)]^{-1}\boldsymbol{y}$ is proportional to the Perron vector $\boldsymbol{q}^*$. This proves the uniqueness of the supporting hyperplane and, furthermore, that $\boldsymbol{y}$ is a positive multiple of $\boldsymbol{y}^*$.

To prove that $\boldsymbol{s}^* \in \partial\mathcal{S}_{\boldsymbol{B}}$ is an exposed point, assume, on the contrary, that there exists some point $\boldsymbol{s}(1) \in \overline{\mathcal{S}}_{\boldsymbol{B}} \setminus \{\boldsymbol{s}^*\}$ such that $\boldsymbol{y}^{*\prime}(\boldsymbol{s}(1) - \boldsymbol{s}^*) = 0$. We will derive contradictory assertions on how far one can follow the line through the points $\boldsymbol{s}^*$ and $\boldsymbol{s}(1)$ without leaving the boundary $\partial\mathcal{S}_{\boldsymbol{B}}$. Thus, let

$$\boldsymbol{s}(\lambda) = \lambda\boldsymbol{s}(1) + (1 - \lambda)\boldsymbol{s}^* \quad \text{and} \quad \boldsymbol{S}(\lambda) = \text{diag}(\boldsymbol{s}(\lambda)).$$

Then the whole segment $\{\boldsymbol{s}(\lambda) \mid 0 \leq \lambda \leq 1\}$ is contained in $\partial\mathcal{S}_{\boldsymbol{B}}$.

Let $\lambda_1 \geq 1$ be the maximum value with

$$\{\boldsymbol{s}(\lambda) \mid 0 \leq \lambda \leq \lambda_1\} \subset \partial\mathcal{S}_{\boldsymbol{B}}$$

and let $\lambda_2 \geq \lambda_1$ be the maximum value such that

$$\{\boldsymbol{s}(\lambda) \mid 0 \leq \lambda < \lambda_2\}$$

remains in the positive orthant $\{\boldsymbol{s} \mid \boldsymbol{s} > \boldsymbol{0}\}$. Note that since $\boldsymbol{y}^* > 0$, the half-line $\{\boldsymbol{s}(\lambda) \mid 0 \leq \lambda\}$ actually leaves the positive orthant. Note also that $\lambda_1 < \lambda_2$, because $\partial\mathcal{S}_{\boldsymbol{B}}$ is a subset of the positive orthant. By (9), $\det(\boldsymbol{S}(\lambda)^{-1}\boldsymbol{B} - \boldsymbol{I}) = 0$ for all $\lambda \in [0, 1]$. But the determinant is a rational function of λ, and so

$$\det(\boldsymbol{S}(\lambda)^{-1}\boldsymbol{B} - \boldsymbol{I}) = 0, \qquad \text{for all } \lambda \in [0, \lambda_2). \tag{13}$$

Since $\rho(\boldsymbol{S}(\lambda_1)^{-1}\boldsymbol{B}) = 1$ is an algebraically simple eigenvalue of $\boldsymbol{S}(\lambda_1)^{-1}\boldsymbol{B}$ and all the eigenvalues of $\boldsymbol{S}(\lambda)^{-1}\boldsymbol{B}$ depend continuously on λ for $0 < \lambda < \lambda_2$, there exist a neighborhood $U \subset \mathbb{C}$ of 1 and $0 < \varepsilon < \frac{1}{2}(\lambda_2 - \lambda_1)$ such that for all $\lambda \in [\lambda_1, \lambda_1 + \varepsilon]$, $\boldsymbol{S}(\lambda)^{-1}\boldsymbol{B}$ has exactly one eigenvalue in U and $\rho(\boldsymbol{S}(\lambda)^{-1}\boldsymbol{B}) \in U \cap \mathbb{R}$. Thus, by (13), $\rho(\boldsymbol{S}(\lambda)^{-1}\boldsymbol{B}) = 1$ and so $\boldsymbol{s}(\lambda) \in \partial\mathcal{S}_{\boldsymbol{B}}$ for all $\lambda \in [\lambda_1, \lambda_1 + \varepsilon]$. This contradicts the definition of λ_1. □

IV. ε-Optimal Power Allocation Without Power Constraints

Let $\boldsymbol{t} > \boldsymbol{0}$ be an infeasible demand profile. That is, there does not exist a power allocation $\boldsymbol{p} > \boldsymbol{0}$ such that the corresponding effective spreading gains $\sigma_i(\boldsymbol{p})$ satisfy $\sigma_i(\boldsymbol{p}) = t_i$ simultaneously for all $i = 1, \ldots, n$. From the perspective of user i, a power allocation should be chosen such that the difference $\sigma_i(\boldsymbol{p}) - t_i$ becomes as small as possible. This is important, e.g., for data applications whenever instead of suboptimal time multiplexing a best effort power balancing point is used. A fair compromise between the conflicting interests of the individual users can be achieved by choosing $\boldsymbol{p}$ as to minimize $\max_{1\leq i\leq n} \sigma_i(\boldsymbol{p}) - t_i$, or somewhat more generally, to minimize

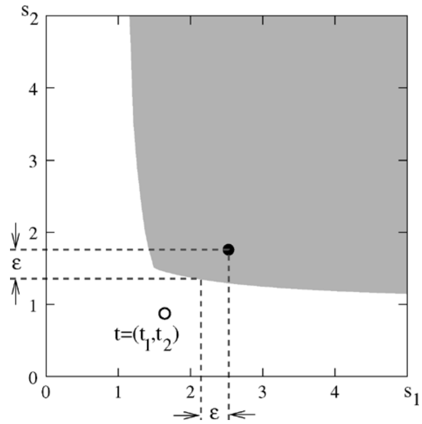

Fig. 1. An ε-minimax power allocation (filled circle) for the infeasible user demand $\boldsymbol{t}$ (empty circle) in two dimensions.

$\max_{1\le i\le n} w_i\left\{\sigma_i(\boldsymbol{p})-t_i\right\}$. Here, $\boldsymbol{w}=(w_1,\ldots,w_n)'>\boldsymbol{0}$ is a given weight vector, where w_i reflects the relative importance attached to user i. It turns out that in the absence of power constraints there does not exist a power allocation that is optimal in the sense indicated, but there are allocations that come arbitrarily close to the optimum. By allowing a maximum deviation ε from the optimum we arrive at the concept of ε-minimax strategies, which is visualized for $n=2$ and constant weights in Fig. 1.

Let $\varepsilon>0$ be fixed. A power allocation $\overline{\boldsymbol{p}}>\boldsymbol{0}$ is said to be an ε *-minimax allocation* for a given infeasible demand profile $\boldsymbol{t}>\boldsymbol{0}$ and weight vector $\boldsymbol{w}>\boldsymbol{0}$ if

$$\max_{1\le i\le n} w_i\left\{\sigma_i(\overline{\boldsymbol{p}})-t_i\right\}\le \inf_{\boldsymbol{p}>\boldsymbol{0}}\max_{1\le i\le n} w_i\left\{\sigma_i(\boldsymbol{p})-t_i\right\}+\varepsilon.$$

The geometric framework developed in Sections II and III allows to construct ε-minimax allocations and to describe their behavior as $\varepsilon\searrow 0$. For this purpose, we need the following well-known minimax representation of the spectral radius (see, e.g., [11, Corollary 8.1.31, p. 493]). For any nonnegative irreducible matrix $\boldsymbol{C}=(c_{ij})_{i,j=1}^n$

$$\rho(\boldsymbol{C})=\min_{\boldsymbol{p}>\boldsymbol{0}}\max_{1\le i\le n}\frac{1}{p_i}\sum_{j=1}^n c_{ij}p_j. \tag{14}$$

Let $\boldsymbol{T}=\operatorname{diag}(\boldsymbol{t})$ and $\boldsymbol{W}=\operatorname{diag}(\boldsymbol{w})$.

Theorem 1: Let μ be the largest real eigenvalue of $\boldsymbol{W}(\boldsymbol{B}-\boldsymbol{T})$ and let $\boldsymbol{q}$ be a corresponding eigenvector normalized so that $\sum_{i=1}^n q_i=1$. Then $\boldsymbol{q}>0$, and $\alpha\boldsymbol{q}$ is an ε-minimax power allocation for $\boldsymbol{t}$ and $\boldsymbol{w}$ if and only if

$$\alpha\ge\frac{1}{\varepsilon}\max\left\{\frac{w_1\tau_1}{q_1},\ldots,\frac{w_n\tau_n}{q_n}\right\}. \tag{15}$$

If for every $\varepsilon>0$, $\boldsymbol{p}(\varepsilon)$ is any ε-minimax power allocation, then as $\varepsilon\searrow 0$

$$p_i(\varepsilon)\to\infty,\quad \frac{p_i(\varepsilon)}{\sum_{j=1}^n p_j(\varepsilon)}\to q_i,\quad w_i\left\{\sigma_i(\boldsymbol{p}(\varepsilon))-t_i\right\}\to\mu \tag{16}$$

$i=1,\ldots,n$, and moreover

$$\sigma_i(\boldsymbol{p}(\varepsilon))\to\frac{1}{q_i}\sum_{j\ne i}b_{ij}q_j,\qquad i=1,\ldots,n.$$

Proof: First we determine the minimax value of the allocation problem. By (8)

$$\begin{aligned}
&\inf_{\boldsymbol{p}>\boldsymbol{0}}\max_{1\le i\le n} w_i\left\{\sigma_i(\boldsymbol{p})-t_i\right\}\\
&\quad=\inf_{\boldsymbol{p}>\boldsymbol{0}}\max_{1\le i\le n} w_i\left\{\frac{1}{p_i}\Big(\tau_i+\sum_{j\ne i}b_{ij}p_j\Big)-t_i\right\}\\
&\quad=\min_{\boldsymbol{p}>\boldsymbol{0}}\max_{1\le i\le n} w_i\left\{\frac{1}{p_i}\left(\sum_{j\ne i}b_{ij}p_j\right)-t_i\right\}\\
&\quad=-\kappa+\min_{\boldsymbol{p}>\boldsymbol{0}}\max_{1\le i\le n}\frac{1}{p_i}\sum_{j=1}^n c_{ij}p_j
\end{aligned}$$

where $\kappa=\max\{w_1t_1,\ldots,w_nt_n\}$ and $\boldsymbol{C}=\boldsymbol{W}(\boldsymbol{B}-\boldsymbol{T})+\kappa\boldsymbol{I}$. The definition of κ ensures that $\boldsymbol{C}$ is nonnegative, and as $\boldsymbol{B}$ is irreducible, so is $\boldsymbol{C}$. If $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of $\boldsymbol{W}(\boldsymbol{B}-\boldsymbol{T})$, then the eigenvalues of $\boldsymbol{C}$ are $\lambda_1+\kappa,\ldots,\lambda_n+\kappa$, the eigenvectors remaining unchanged. Hence, $\rho(\boldsymbol{C})=\mu+\kappa$ and $\boldsymbol{q}>\boldsymbol{0}$. It now follows from (14) that

$$\inf_{\boldsymbol{p}>\boldsymbol{0}}\max_{1\le i\le n} w_i\left\{\sigma_i(\boldsymbol{p})-t_i\right\}=\mu.$$

For the power allocation $\alpha\boldsymbol{q}$ one obtains from the equation $\boldsymbol{W}(\boldsymbol{B}-\boldsymbol{T})\boldsymbol{q}=\mu\boldsymbol{q}$ that for $i=1,\ldots,n$

$$w_i\left\{\sigma_i(\alpha\boldsymbol{q})-t_i\right\}=\frac{w_i\tau_i}{\alpha q_i}+\frac{w_i}{q_i}\left(\sum_{j\ne i}b_{ij}q_j\right)-w_it_i=\frac{w_i\tau_i}{\alpha q_i}+\mu.$$

This shows that $\alpha\boldsymbol{q}$ is ε-minimax if and only if α satisfies (15).

Now suppose that for every $\varepsilon>0$, $\boldsymbol{p}(\varepsilon)>0$ is an arbitrary ε-minimax allocation, so that

$$w_i\left\{\sigma_i(\boldsymbol{p}(\varepsilon))-t_i\right\}\le\mu+\varepsilon,\qquad i=1,\ldots,n,\ \varepsilon>0. \tag{17}$$

It will be shown that $\boldsymbol{\sigma}(\boldsymbol{p}(\varepsilon))\to\boldsymbol{s}^*$ as $\varepsilon\searrow 0$, where $\boldsymbol{s}^*=\boldsymbol{Q}^{-1}\boldsymbol{B}\boldsymbol{q}$ and $\boldsymbol{Q}=\operatorname{diag}(\boldsymbol{q})$. By Proposition 3 a), $\boldsymbol{s}^*\in\partial\mathcal{S}_{\boldsymbol{B}}$, and $\boldsymbol{s}^*$ can be written as

$$\boldsymbol{s}^*=\boldsymbol{Q}^{-1}(\boldsymbol{T}\boldsymbol{q}+\mu\boldsymbol{W}^{-1}\boldsymbol{q})=\boldsymbol{t}+\mu(w_1^{-1},\ldots,w_n^{-1})'. \tag{18}$$

According to Proposition 4, there exists $\boldsymbol{y}>\boldsymbol{0}$ such that $0\le\boldsymbol{y}'(\boldsymbol{s}-\boldsymbol{s}^*)$ for all $\boldsymbol{s}\in\overline{\mathcal{S}_{\boldsymbol{B}}}$ with equality if and only if $\boldsymbol{s}=\boldsymbol{s}^*$. Hence, by (17)

$$0\le\boldsymbol{y}'(\boldsymbol{\sigma}(\boldsymbol{p}(\varepsilon))-\boldsymbol{s}^*)\le\sum_{i=1}^n y_i\left(\frac{\mu+\varepsilon}{w_i}+t_i\right)-\boldsymbol{y}'\boldsymbol{s}^*$$

and, by (18), the expression on the right-hand side tends to 0 as $\varepsilon\searrow 0$. Thus, every convergent subnet of $\{\boldsymbol{\sigma}(\boldsymbol{p}(\varepsilon));0<\varepsilon\le 1\}$ must converge to $\boldsymbol{s}^*$, and since the net is bounded, it follows that $\boldsymbol{\sigma}(\boldsymbol{p}(\varepsilon))\to\boldsymbol{s}^*$. This and (18) prove the third assertion in (16) and the first two follow from Proposition 3 b). □

By Theorem 1, the remaining gap between the demand profile $\boldsymbol{t}$ and the realized spreading gain of any ε-minimax power allocation is μ, the largest real eigenvalue of $\boldsymbol{W}(\boldsymbol{B}-\boldsymbol{T})$, which always exists. The corresponding eigenvector $\boldsymbol{q}$ determines the limit direction of any sequence of power allocations achieving this gap.

Next we aim at minimizing a weighted mean of $\sigma_i(\boldsymbol{p})$ in the following sense. Let $\varepsilon>0$ be fixed. A power allocation $\overline{\boldsymbol{p}}>\boldsymbol{0}$ is said to be an ε *-Bayes allocation* with respect to a given weight vector $\boldsymbol{w} > \mathbf{0}$ if

$$\sum_{i=1}^{n} w_i \sigma_i(\overline{\boldsymbol{p}}) \leq \inf_{\boldsymbol{p}>\mathbf{0}} \sum_{i=1}^{n} w_i \sigma_i(\boldsymbol{p}) + \varepsilon.$$

Bayes allocations also realize a certain best effort power balancing scheme. In contrast to the previous minimax criterion, not the worst case, but a weighted average of user interests is employed as the benefit criterion.

Theorem 2: There is a unique $\boldsymbol{p}^* > \mathbf{0}$ such that

$$\boldsymbol{W}\boldsymbol{P}^{*-1}\boldsymbol{B}\boldsymbol{p}^* = \boldsymbol{P}^*\boldsymbol{B}'\boldsymbol{P}^{*-1}\boldsymbol{w}, \qquad \sum_{i=1}^{n} p_i^* = 1 \tag{19}$$

where $\boldsymbol{P}^* = \operatorname{diag}(\boldsymbol{p}^*)$. The power allocation $\alpha\boldsymbol{p}^*$ is an ε-Bayes allocation with respect to $\boldsymbol{w}$ if and only if

$$\alpha \geq \frac{1}{\varepsilon} \sum_{i=1}^{n} \frac{w_i \tau_i}{p_i^*}. \tag{20}$$

If for every $\varepsilon > 0$, $\boldsymbol{p}(\varepsilon)$ is any ε-Bayes power allocation, then for $i = 1, \ldots, n$, as $\varepsilon \searrow 0$

$$p_i(\varepsilon) \to \infty, \ \ \frac{p_i(\varepsilon)}{\sum_{j=1}^{n} p_j(\varepsilon)} \to p_i^*, \ \ \sigma_i(\boldsymbol{p}(\varepsilon)) \to \frac{1}{p_i^*} \sum_{j \neq i} b_{ij} p_j^*. \tag{21}$$

Proof: It will be first shown that (19) does have a positive solution. Since $\boldsymbol{w} > \mathbf{0}$, the linear function $\boldsymbol{s} \mapsto \boldsymbol{w}'\boldsymbol{s}$ attains its minimum over $\overline{\mathcal{S}}_{\boldsymbol{B}}$ at some point $\boldsymbol{s}^* \in \partial\mathcal{S}_{\boldsymbol{B}}$. Thus, $\{\boldsymbol{s} \mid \boldsymbol{w}'(\boldsymbol{s} - \boldsymbol{s}^*) = 0\}$ is a supporting hyperplane to $\overline{\mathcal{S}}_{\boldsymbol{B}}$ at $\boldsymbol{s}^*$. Now set $\boldsymbol{S}^* = \operatorname{diag}(\boldsymbol{s}^*)$ and let $\boldsymbol{p}^* > \mathbf{0}$ and $\boldsymbol{q}^* > \mathbf{0}$ be the Perron vectors of $\boldsymbol{S}^{*-1}\boldsymbol{B}$ and $\boldsymbol{S}^{*-1}\boldsymbol{B}'$, respectively. Thus, $\boldsymbol{B}\boldsymbol{p}^* = \boldsymbol{S}^*\boldsymbol{p}^*$ and $\boldsymbol{B}'\boldsymbol{q}^* = \boldsymbol{S}^*\boldsymbol{q}^*$. By Proposition 4, the supporting hyperplane to $\overline{\mathcal{S}}_{\boldsymbol{B}}$ at $\boldsymbol{s}^*$ is unique and the normal is proportional to $\boldsymbol{P}^*\boldsymbol{q}^*$. Hence, $\boldsymbol{P}^*\boldsymbol{q}^* = c\boldsymbol{w}$ for some $c \neq 0$. It follows that

$$\begin{aligned} \boldsymbol{P}^*\boldsymbol{B}'\boldsymbol{P}^{*-1}\boldsymbol{w} &= \frac{1}{c}\boldsymbol{P}^*\boldsymbol{B}'\boldsymbol{q}^* = \frac{1}{c}\boldsymbol{P}^*\boldsymbol{S}^*\boldsymbol{q}^* \\ &= \frac{1}{c}\boldsymbol{S}^*\boldsymbol{P}^*\boldsymbol{q}^* = \boldsymbol{S}^*\boldsymbol{w}. \end{aligned}$$

On the other hand

$$\boldsymbol{W}\boldsymbol{P}^{*-1}\boldsymbol{B}\boldsymbol{p}^* = \boldsymbol{W}\boldsymbol{P}^{*-1}\boldsymbol{S}^*\boldsymbol{p}^* = \boldsymbol{W}\boldsymbol{s}^* = \boldsymbol{S}^*\boldsymbol{w}$$

proving the existence of a positive solution of (19).

Now let $\boldsymbol{p}^*$ be any solution of (19) and define $\boldsymbol{s}^* = \boldsymbol{P}^{*-1}\boldsymbol{B}\boldsymbol{p}^*$ and $\boldsymbol{q}^* = \boldsymbol{P}^{*-1}\boldsymbol{w}$. Then, by (19)

$$\begin{aligned} \boldsymbol{B}'\boldsymbol{q}^* &= \boldsymbol{B}'\boldsymbol{P}^{*-1}\boldsymbol{w} = \boldsymbol{P}^{*-1}\boldsymbol{W}\boldsymbol{P}^{*-1}\boldsymbol{B}\boldsymbol{p}^* = \boldsymbol{P}^{*-1}\boldsymbol{W}\boldsymbol{s}^* \\ &= \boldsymbol{S}^*\boldsymbol{P}^{*-1}\boldsymbol{w} = \boldsymbol{S}^*\boldsymbol{q}^*. \end{aligned}$$

Therefore, $\boldsymbol{q}^*$ is proportional to the Perron vector of $\boldsymbol{S}^{*-1}\boldsymbol{B}'$. Besides, $\boldsymbol{p}^*$ is the Perron vector of $\boldsymbol{S}^{*-1}\boldsymbol{B}$. It thus follows from Proposition 4 that $\boldsymbol{w}'(\boldsymbol{s} - \boldsymbol{s}^*) > 0$ for all $\boldsymbol{s} \in \overline{\mathcal{S}}_{\boldsymbol{B}} \setminus \{\boldsymbol{s}^*\}$. In particular, $\inf_{\boldsymbol{p}>\mathbf{0}} \sum_{i=1}^{n} w_i \sigma_i(\boldsymbol{p}) = \boldsymbol{w}'\boldsymbol{s}^*$. For the candidate $\alpha\boldsymbol{p}^*$ one has

$$\sum_{i=1}^{n} w_i \sigma_i(\alpha\boldsymbol{p}^*) = \boldsymbol{w}'\boldsymbol{s}^* + \sum_{i=1}^{n} \frac{w_i \tau_i}{\alpha p_i^*}.$$

Hence $\alpha\boldsymbol{p}^*$ is ε-Bayes if and only if (20) holds. It is also easy to see that if for every $\varepsilon > 0$, $\boldsymbol{p}(\varepsilon)$ is an arbitrary ε-Bayes power allocation, then $\boldsymbol{\sigma}(\boldsymbol{p}(\varepsilon)) \to \boldsymbol{s}^*$ as $\varepsilon \searrow 0$. In view of Proposition 3 b), (21) follows immediately. In particular, the solution $\boldsymbol{p}^*$ of (19) turns out to be unique. □

V. Capacity Regions With Power Constraints

In this section, we extend the geometric results of Section III to capacity regions which arise when the transmit power of each user is bounded. If $\hat{\boldsymbol{p}} = (\hat{p}_1, \ldots, \hat{p}_n)' > \mathbf{0}$ is a fixed vector of maximum transmit powers, the corresponding capacity region is given by

$$\mathcal{S}_{\boldsymbol{B}}(\hat{\boldsymbol{p}}) = \{\boldsymbol{\sigma}(\boldsymbol{p}) \mid \mathbf{0} < \boldsymbol{p} \leq \hat{\boldsymbol{p}}\}$$

where $\boldsymbol{\sigma}(\boldsymbol{p}) = \boldsymbol{P}^{-1}(\boldsymbol{\tau} + \boldsymbol{B}\boldsymbol{p})$ and $\boldsymbol{P} = \operatorname{diag}(\boldsymbol{p})$.

As a central result of this section, the convexity of the set $\mathcal{S}_{\boldsymbol{B}}(\hat{\boldsymbol{p}})$ is obtained. This forms the basis for various strategies aiming at optimizing $\boldsymbol{p}$. A first application is given in the next section. The result itself seems reasonable: if two demand profiles can be served by the network subject to a given power constraint, then one would expect that a convex combination of these profiles can be served subject to the same constraint. However, a direct verification appears to be rather difficult even when $n = 3$. The arguments used in [4] to prove convexity in the unconstrained case do not carry over to the present situation.

The key observation is that if one restricts only one component of $\boldsymbol{p}$, then the resulting capacity region is basically the same as the closure of a certain capacity region without contraints associated with a suitably chosen path loss pattern. This will be shown in two steps. First, we focus on the relation between the boundaries of both regions. Then, using the results already proved for the unrestricted case, we extend this relation to the sets themselves.

Thus, consider, for each $k = 1, \ldots, n$, the manifold

$$\mathcal{S}^{(k)} = \mathcal{S}_{\boldsymbol{B}}^{(k)}(\hat{p}_k) = \{\boldsymbol{\sigma}(\boldsymbol{p}) \mid \mathbf{0} < \boldsymbol{p} \text{ and } p_k \leq \hat{p}_k\}$$

and the surface

$$\mathcal{R}^{(k)} = \mathcal{R}_{\boldsymbol{B}}^{(k)}(\hat{p}_k) = \{\boldsymbol{\sigma}(\boldsymbol{p}) \mid \mathbf{0} < \boldsymbol{p} \text{ and } p_k = \hat{p}_k\}.$$

Obviously, $\mathcal{R}^{(k)} \subset \partial\mathcal{S}^{(k)}$. To see that both sets actually coincide, first note that

$$\partial\mathcal{S}^{(k)} = \left[(\partial\mathcal{S}^{(k)}) \cap \mathcal{S}_{\boldsymbol{B}}\right] \cup \left[(\partial\mathcal{S}^{(k)}) \cap \partial\mathcal{S}_{\boldsymbol{B}}\right].$$

But $\mathcal{S}^{(k)}$ and $\mathcal{S}_{\boldsymbol{B}}$ do not have common boundary points. For if $\boldsymbol{s}^* \in \partial\mathcal{S}^{(k)}$, then there exists a sequence $\{\boldsymbol{s}(j)\}_{j=1}^{\infty} \subset \mathcal{S}^{(k)}$ with $\boldsymbol{s}(j) \to \boldsymbol{s}^*$ and $\sup_j \pi_k(\boldsymbol{s}(j)) \leq \hat{p}_k$. By Proposition 3 b), $\boldsymbol{s}^* \notin \partial\mathcal{S}_{\boldsymbol{B}}$. Thus, $\partial\mathcal{S}^{(k)} = \left(\partial\mathcal{S}^{(k)}\right) \cap \mathcal{S}_{\boldsymbol{B}}$, which is just the boundary of $\mathcal{S}^{(k)}$ in $\mathcal{S}_{\boldsymbol{B}}$ with respect to the relative topology. The map $\boldsymbol{\sigma}$ is a homeomorphism from $\mathcal{P} = \{\boldsymbol{p} \mid \boldsymbol{p} > \mathbf{0}\}$ onto $\mathcal{S}_{\boldsymbol{B}}$. Hence, the boundary of $\mathcal{S}^{(k)}$ in $\mathcal{S}_{\boldsymbol{B}}$ is

$$\boldsymbol{\sigma}((\partial\{\boldsymbol{p} > \mathbf{0} \mid p_k \leq \hat{p}_k\}) \cap \mathcal{P}) = \mathcal{R}^{(k)}.$$

Altogether

$$\mathcal{R}^{(k)} = \partial\mathcal{S}^{(k)}. \tag{22}$$

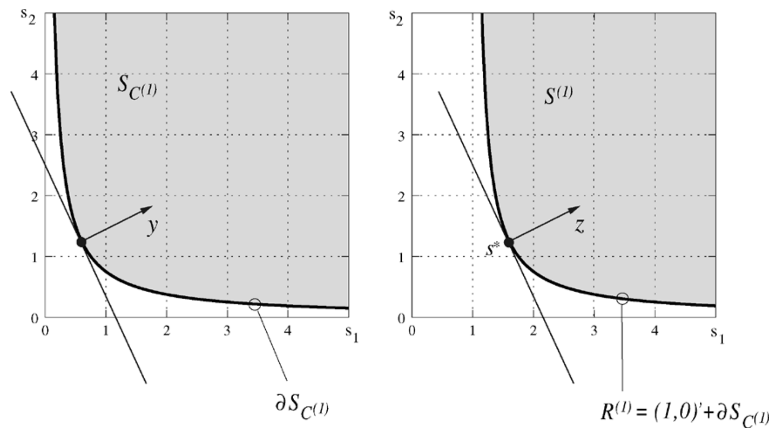

Fig. 2. A graphical overview of the proof principles in two dimensions. Concrete values are $n = 2, a_{21}/a_{11} = a_{12}/a_{22} = 0.5, \tau_i/\hat{p}_i = 1, i = 1, 2$.

The link between $\partial\mathcal{S}^{(k)}$, the boundary of the capacity region with a single power constraint $p_k \le \hat{p}_k$, and the boundary of a certain capacity region without power constraints can be established through the following representation of $\boldsymbol{\sigma}$, which is valid only on the boundary $\mathcal{R}^{(k)}$

$$\boldsymbol{\sigma}(\boldsymbol{p}) = \frac{\tau_k}{\hat{p}_k}\boldsymbol{e}_k + \boldsymbol{P}^{-1}\boldsymbol{C}^{(k)}\boldsymbol{p}, \qquad \text{for all } \boldsymbol{p} > \boldsymbol{0} \text{ with } p_k = \hat{p}_k \tag{23}$$

where $\boldsymbol{e}_k \in \mathbb{R}^n$ denotes the kth unit vector, and $\boldsymbol{C}^{(k)} = (c_{ij}^{(k)})_{ij=1}^n$ is defined by

$$c_{ij}^{(k)} = \begin{cases} b_{ik} + \tau_i/\hat{p}_k, & \text{if } i \neq k \text{ and } j = k \\ b_{ij}, & \text{otherwise.} \end{cases}$$

That is, $\boldsymbol{C}^{(k)}$ is the matrix obtained from $\boldsymbol{B}$ by adding the vector $\hat{p}_k^{-1}(\tau_1, \dots, \tau_{k-1}, 0, \tau_{k+1}, \dots, \tau_n)'$ to the kth column. In particular, $\boldsymbol{C}^{(k)}$ is nonnegative and irreducible, so that all the results of Section III apply with $\boldsymbol{B}$ replaced by $\boldsymbol{C}^{(k)}$. Equation (23) may be seen as follows. If $p_k = \hat{p}_k$, then

$$\sigma_k(\boldsymbol{p}) = \frac{1}{p_k}\Big(\tau_k + \sum_{j\neq k} b_{kj}\, p_j\Big) = \frac{\tau_k}{\hat{p}_k} + \frac{1}{p_k}\sum_{j\neq k} c_{kj}^{(k)}\, p_j$$

and for $i \neq k$

$$\begin{aligned}\sigma_i(\boldsymbol{p}) &= \frac{1}{p_i}\Big(\tau_i + \sum_{j\neq i} b_{ij}\, p_j\Big)\\ &= \frac{1}{p_i}\Big(\big(\frac{\tau_i}{\hat{p}_k} + b_{ik}\big)p_k + \sum_{j\neq k,i} b_{ij}\, p_j\Big) = \frac{1}{p_i}\Big(\sum_{j\neq i} c_{ij}^{(k)}\, p_j\Big).\end{aligned}$$

Now by (23)

$$\mathcal{R}^{(k)} = (\tau_k/\hat{p}_k)\boldsymbol{e}_k + \{\boldsymbol{P}^{-1}\boldsymbol{C}^{(k)}\boldsymbol{p} \mid \boldsymbol{p} > \boldsymbol{0}, p_k = \hat{p}_k\}.$$

But the set $\{\boldsymbol{P}^{-1}\boldsymbol{C}^{(k)}\boldsymbol{p} \mid \boldsymbol{p} > \boldsymbol{0}, p_k = \hat{p}_k\}$ obviously remains unchanged when the power constraint $p_k = \hat{p}_k$ is removed. In view of Proposition 3 a), $\{\boldsymbol{P}^{-1}\boldsymbol{C}^{(k)}\boldsymbol{p} \mid \boldsymbol{p} > \boldsymbol{0}\}$ is the boundary of the whole unrestricted capacity region for $\boldsymbol{C}^{(k)}$. Recalling (22), we have thus proved the following representations of the boundary of $\mathcal{S}^{(k)}$.

Proposition 5: For $k = 1, \dots, n$

$$\partial\mathcal{S}^{(k)} = \mathcal{R}^{(k)} = \frac{\tau_k}{\hat{p}_k}\boldsymbol{e}_k + \partial\mathcal{S}_{\boldsymbol{C}^{(k)}}.$$

Proposition 5 says that the capacity region under path loss pattern $\boldsymbol{B}$ for n mobiles with the power of the kth fixed to $\hat{p}_k$ coincides with the boundary of the unconstrained capacity region under path loss pattern $\boldsymbol{C}^{(k)}$ shifted by a constant vector.

Proposition 6: Each of the sets $\mathcal{S}^{(k)}, k = 1, \dots, n$ is a closed convex subset of $\mathbb{R}_+^n$, and for every $\boldsymbol{s}^* \in \partial\mathcal{S}^{(k)}$ there even exists some $\boldsymbol{z} = \boldsymbol{z}(\boldsymbol{s}^*) > \boldsymbol{0}$ such that

$$\boldsymbol{z}'(\boldsymbol{s} - \boldsymbol{s}^*) > 0, \qquad \text{for all } \boldsymbol{s} \in \mathcal{S}^{(k)} \setminus \{\boldsymbol{s}^*\}. \tag{24}$$

Proof: In view of Proposition 5, $\partial\mathcal{S}^{(k)} = \mathcal{R}^{(k)} \subset \mathcal{S}^{(k)}$, showing that $\mathcal{S}^{(k)}$ is closed. To prove (24), fix $\boldsymbol{s}^* \in \partial\mathcal{S}^{(k)}$. It follows from Propositions 5 and 4 that there exists $\boldsymbol{z} = \boldsymbol{z}(\boldsymbol{s}^*) > \boldsymbol{0}$ such that the inequality in (24) holds for all $\boldsymbol{s} \in \partial\mathcal{S}^{(k)} \setminus \{\boldsymbol{s}^*\}$. To show that the inequality continues to hold on the interior of $\mathcal{S}^{(k)}$, let $\boldsymbol{s}^{(1)}$ be an arbitrary interior point and set $\boldsymbol{p} = \boldsymbol{\pi}(\boldsymbol{s}^{(1)})$. Then $0 < p_k < \hat{p}_k$, and multiplying $\boldsymbol{p}$ by $\alpha = \hat{p}_k/p_k > 1$ leads to the point $\boldsymbol{s}^{(2)} = \boldsymbol{\sigma}(\alpha\boldsymbol{p}) \in \partial\mathcal{S}^{(k)}$. For $i = 1, \dots, n$

$$s_i^{(2)} = \frac{\tau_i}{\alpha p_i} + \frac{1}{p_i}\sum_{j\neq i} b_{ij} p_j < \frac{\tau_i}{p_i} + \frac{1}{p_i}\sum_{j\neq i} b_{ij} p_j = s_i^{(1)}.$$

Hence,

$$\boldsymbol{z}'(\boldsymbol{s}^{(1)} - \boldsymbol{s}^*) > \boldsymbol{z}'(\boldsymbol{s}^{(2)} - \boldsymbol{s}^*) \ge 0$$

proving (24). Thus, through every boundary point of $\mathcal{S}^{(k)}$ there passes a supporting hyperplane to $\mathcal{S}^{(k)}$. This implies that $\mathcal{S}^{(k)}$ is convex, provided that the interior of $\mathcal{S}^{(k)}$ is not empty, see [13, p. 21]. The existence of interior points is obvious from Proposition 5. □

Fig. 2 gives a graphical overview of the proof principles so far in $n =$ two dimensions for $k = 1$.

The convexity of $\mathcal{S}_{\boldsymbol{B}}(\hat{\boldsymbol{p}})$ is an immediate consequence of Proposition 6 since $\mathcal{S}_{\boldsymbol{B}}(\hat{\boldsymbol{p}}) = \bigcap_{k=1}^n \mathcal{S}^{(k)}$, as we state in the following theorem.

Theorem 3: For any given power constraint $\boldsymbol{0} < \boldsymbol{p} \le \hat{\boldsymbol{p}}$, the corresponding capacity region $\mathcal{S}_{\boldsymbol{B}}(\hat{\boldsymbol{p}})$ is a closed convex subset of $\mathbb{R}_+^n$.

Fig. 3 shows the capacity region for $n =$ two mobiles. It is assumed that $b_{12} = a_{21}/a_{11} = b_{21} = a_{12}/a_{22} = 0.5$, i.e.,

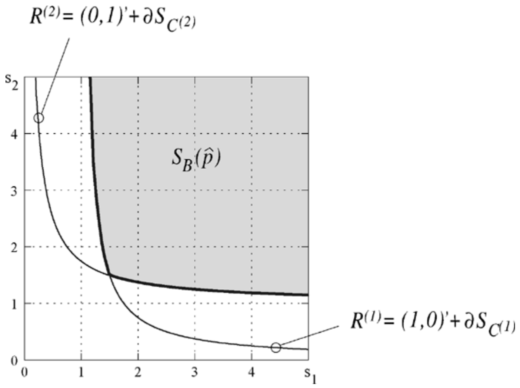

Fig. 3. The capacity region for $n = 2, a_{21}/a_{11} = a_{12}/a_{22} = 0.5, \tau_i/\hat{p}_i = 1, i = 1, 2$.

the path loss to the other mobile's base station is twice as high as to one's own. The ratio $\tau_i/\hat{p}_i = 1$. Recall that the relative background noise is defined as $\tau_i = \tilde{\tau}_{k_i}/a_{ik_i}, i = 1, 2$. Because of symmetry, the sets $\mathcal{S}_{\boldsymbol{C}^{(k)}}$ coincide for both cases $k = 1$ and $k = 2$. The capacity region $\mathcal{S}_{\boldsymbol{B}}(\hat{\boldsymbol{p}})$ is the intersection $\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$, as shown in Fig. 3.

Another consequence of Proposition 6 is that the relation between the boundaries $\partial\mathcal{S}^{(k)}$ and $\partial\mathcal{S}_{\boldsymbol{C}^{(k)}}$ given in Proposition 5 extends to the sets themselves.

Proposition 7: For $k = 1, \dots, n$

$$\mathcal{S}^{(k)} = \frac{\tau_k}{\hat{p}_k}\boldsymbol{e}_k + \overline{\mathcal{S}_{\boldsymbol{C}^{(k)}}}.$$

Proof: A closed convex set which is not an affine set or a closed half of an affine set is the convex hull of its boundary, see [14, p. 166]. Thus, by Proposition 6, $\mathcal{S}^{(k)} = \operatorname{conv}(\partial\mathcal{S}^{(k)})$ and by Proposition 2, $\overline{\mathcal{S}_{\boldsymbol{C}^{(k)}}} = \operatorname{conv}(\partial\mathcal{S}_{\boldsymbol{C}^{(k)}})$. The assertion is now obvious from Proposition 5. □

As an application of Proposition 7, we establish a result which is intuitively clear but would be intricate to verify directly. It says that if a demand profile can be satisfied subject to a given power constraint, then any less stringent demand profile can be served subject to the same constraint.

Proposition 8: If $\boldsymbol{s} \in \mathcal{S}^{(k)}$ and $\boldsymbol{u} \geq \boldsymbol{s}, \boldsymbol{u} \neq \boldsymbol{s}$, then $\boldsymbol{u} \in \operatorname{int}(\mathcal{S}^{(k)})$. If $\boldsymbol{s} \in \mathcal{S}_{\boldsymbol{B}}(\hat{\boldsymbol{p}})$ and $\boldsymbol{u} \geq \boldsymbol{s}, \boldsymbol{u} \neq \boldsymbol{s}$, then $\boldsymbol{u} \in \operatorname{int}(\mathcal{S}_{\boldsymbol{B}}(\hat{\boldsymbol{p}}))$.

Proof: The second assertion follows from the first since $\mathcal{S}_{\boldsymbol{B}}(\hat{\boldsymbol{p}}) = \bigcap_{k=1}^n \mathcal{S}^{(k)}$. To prove the first, it suffices by Proposition 7 to show that if $\boldsymbol{x} \in \overline{\mathcal{S}_{\boldsymbol{C}^{(k)}}}$ and $\boldsymbol{y} \geq \boldsymbol{x}, \boldsymbol{y} \neq \boldsymbol{x}$, then $\boldsymbol{y} \in \operatorname{int}(\mathcal{S}_{\boldsymbol{C}^{(k)}})\, (= \mathcal{S}_{\boldsymbol{C}^{(k)}})$. Write $\boldsymbol{X} = \operatorname{diag}(\boldsymbol{x})$ and $\boldsymbol{Y} = \operatorname{diag}(\boldsymbol{y})$. Suppose first that $\boldsymbol{x} \in \mathcal{S}_{\boldsymbol{C}^{(k)}}$, that is, $\rho(\boldsymbol{X}^{-1}\boldsymbol{C}^{(k)}) < 1$. The monotonicity of the spectral radius gives that

$$\rho(\boldsymbol{Y}^{-1}\boldsymbol{C}^{(k)}) \leq \rho(\boldsymbol{X}^{-1}\boldsymbol{C}^{(k)})$$

so that $\boldsymbol{y} \in \mathcal{S}_{\boldsymbol{C}^{(k)}}$. Now suppose that $\boldsymbol{x} \in \partial\mathcal{S}_{\boldsymbol{C}^{(k)}}$ and set $\boldsymbol{Z} = \frac{1}{2}(\boldsymbol{X} + \boldsymbol{Y})$. The monotonicity of ρ yields only that

$$\rho(\boldsymbol{Y}^{-1}\boldsymbol{C}^{(k)}) \leq \rho(\boldsymbol{Z}^{-1}\boldsymbol{C}^{(k)}) \leq \rho(\boldsymbol{X}^{-1}\boldsymbol{C}^{(k)}) = 1.$$

But if $\rho(\boldsymbol{Y}^{-1}\boldsymbol{C}^{(k)}) = 1$, it would follow that the three points $\boldsymbol{x}, \frac{1}{2}(\boldsymbol{x} + \boldsymbol{y}), \boldsymbol{y}$ lie on the boundary of $\mathcal{S}_{\boldsymbol{C}^{(k)}}$. This is impossible, because according to Proposition 4, every boundary point is an extreme point of $\overline{\mathcal{S}_{\boldsymbol{C}^{(k)}}}$. Hence, $\rho(\boldsymbol{Y}^{-1}\boldsymbol{C}^{(k)}) < 1$. □

VI. Optimal Power Allocation With Power Constraints

If power is limited, the capacity region is closed, whereby for any boundary point there is an admissible power allocation. The power assignment which corresponds to the feasible point closest to some given infeasible user demand profile can be realized by the network and is called minimax allocation, with distance measured by the maximum norm. Fig. 1 applies accordingly with $\varepsilon = 0$.

To be more precise, a power allocation $\boldsymbol{p}^* > \boldsymbol{0}$ is said to be a minimax allocation for a given infeasible demand profile $\boldsymbol{t} > \boldsymbol{0}$, a weight vector $\boldsymbol{w} > \boldsymbol{0}$, and a power bound $\hat{\boldsymbol{p}}$ if $\boldsymbol{p}^* \leq \hat{\boldsymbol{p}}$ and

$$\max_{1\leq i\leq n} w_i\{\sigma_i(\boldsymbol{p}^*) - t_i\} = \min_{\boldsymbol{0}<\boldsymbol{p}\leq\hat{\boldsymbol{p}}} \max_{1\leq i\leq n} w_i\{\sigma_i(\boldsymbol{p}) - t_i\}.$$

The weights w_i reflect service priorities to different users.

Theorem 4: For $k = 1, \dots, n$, let $\boldsymbol{D}^{(k)}$ denote the matrix obtained from $\boldsymbol{B}$ by adding the vector $\hat{p}_k^{-1}\boldsymbol{\tau}$ to the kth column of $\boldsymbol{B}$ and let μ_k be the largest real eigenvalue of $\boldsymbol{W}(\boldsymbol{D}^{(k)} - \boldsymbol{T})$. Let $k^* \in \{1, \dots, n\}$ be such that $\mu_{k^*} = \max\{\mu_1, \dots, \mu_n\}$. The unique minimax power allocation for the demand profile $\boldsymbol{t}$ and the weight $\boldsymbol{w}$ subject to $\boldsymbol{p} \leq \hat{\boldsymbol{p}}$ is given by the unique eigenvector $\boldsymbol{p}^*$ of $\boldsymbol{W}(\boldsymbol{D}^{(k^*)} - \boldsymbol{T})$ corresponding to μ_{k^*} normalized so that $p_{k^*}^* = \hat{p}_{k^*}$. Moreover, $w_i\{\sigma_i(\boldsymbol{p}^*) - t_i\} = \mu_{k^*}$ for $i = 1, \dots, n$.

Proof: The proof consists of two steps. First, for each k, the minimax allocation subject to the single constraint $p_k \leq \hat{p}_k$ will be determined. It will then be shown that among these allocations the one corresponding to k^* satisfies all constraints $p_k \leq \hat{p}_k$ simultaneously and so is the sought allocation.

As $\boldsymbol{B}$ is nonnegative and irreducible, so is $\boldsymbol{W}(\boldsymbol{D}^{(k)} - \boldsymbol{T}) + \kappa\boldsymbol{I}_n$, provided κ is chosen sufficiently large. The largest real eigenvalue of that matrix, $\mu_k + \kappa$, is therefore simple and a corresponding eigenvector $\boldsymbol{p}(k) = (p_1(k), \dots, p_n(k))'$ can be chosen so that $\boldsymbol{p}(k) > \boldsymbol{0}$ and $p_k(k) = \hat{p}_k$. Note that $\boldsymbol{p}(k^*) = \boldsymbol{p}^*$ and that $\boldsymbol{p}^*$ is indeed uniquely determined. From the equations $\mu_k\boldsymbol{p}(k) = \boldsymbol{W}(\boldsymbol{D}^{(k)} - \boldsymbol{T})\boldsymbol{p}(k)$ and $\boldsymbol{D}^{(k)}\boldsymbol{p}(k) = \boldsymbol{\tau} + \boldsymbol{B}\boldsymbol{p}(k)$ one obtains that

$$\begin{aligned}\mu_k\boldsymbol{1} &= [\operatorname{diag}(\boldsymbol{p}(k))]^{-1}\boldsymbol{W}(\boldsymbol{D}^{(k)} - \boldsymbol{T})\boldsymbol{p}(k)\\ &= \boldsymbol{W}\left[[\operatorname{diag}(\boldsymbol{p}(k))]^{-1}[\boldsymbol{\tau} + \boldsymbol{B}\boldsymbol{p}(k)] - \boldsymbol{t}\right]\\ &= \boldsymbol{W}[\boldsymbol{\sigma}(\boldsymbol{p}(k)) - \boldsymbol{t}].\end{aligned} \tag{25}$$

Thus,

$$w_i\{\sigma_i(\boldsymbol{p}(k)) - t_i\} = \mu_k, \qquad i = 1, \dots, n. \tag{26}$$

By Proposition 5, $\boldsymbol{s}(k) = \boldsymbol{\sigma}(\boldsymbol{p}(k))$ is a boundary point of $\mathcal{S}^{(k)}$. It therefore follows from Proposition 6 that there exists $\boldsymbol{z} = \boldsymbol{z}(k) > \boldsymbol{0}$ such that $\boldsymbol{z}'(\boldsymbol{s} - \boldsymbol{s}(k)) > 0$ for all $\boldsymbol{s} \in \mathcal{S}^{(k)} \setminus \{\boldsymbol{s}(k)\}$. Hence,

$$\begin{aligned}0 &< \sum_{i=1}^n \frac{z_i}{w_i} w_i(s_i - t_i) - \sum_{i=1}^n \frac{z_i}{w_i} w_i(s_i(k) - t_i)\\ &\leq \left\{\max_{1\leq i\leq n} w_i(s_i - t_i)\right\} \sum_{i=1}^n \frac{z_i}{w_i} - \mu_k \sum_{i=1}^n \frac{z_i}{w_i}\end{aligned}$$

so that

$$\max_{1\leq i\leq n} w_i(s_i - t_i) > \mu_k \text{ for all } \boldsymbol{s} \in \mathcal{S}^{(k)} \setminus \{\boldsymbol{s}(k)\}. \tag{27}$$

It is thus proved that $\boldsymbol{p}(k)$ is a minimax power allocation subject to the constraint $p_k \le \hat{p}_k$, but $\boldsymbol{s}(k)$ need not belong to $\mathcal{S}_{\boldsymbol{B}}(\hat{\boldsymbol{p}})$. By (25), $\boldsymbol{s}(k) = \boldsymbol{t} + \mu_k \boldsymbol{W}^{-1}\mathbf{1} \in \mathcal{S}^{(k)}$ and since $\mu_{k^*} \ge \mu_k$, $\boldsymbol{s}(k^*) \ge \boldsymbol{s}(k)$. Therefore, by Proposition 8, $\boldsymbol{s}(k^*) \in \mathcal{S}^{(k)}$ for all k, and so $\boldsymbol{s}(k^*) \in \bigcap_{k=1}^{n} \mathcal{S}^{(k)} = \mathcal{S}_{\boldsymbol{B}}(\hat{\boldsymbol{p}})$. It now follows from (26) and (27) that $\boldsymbol{p}(k^*) = \boldsymbol{p}^*$ is the minimax power allocation. Note finally that k^* need not be unique, but the point $\boldsymbol{s}(k^*) = \boldsymbol{t} + \mu_{k^*}\boldsymbol{W}^{-1}\mathbf{1}$ is independent of the choice of k^*, so that the minimax power allocation $\boldsymbol{\pi}(\boldsymbol{s}(k^*))$ is indeed unique. □

In Theorem 4 it is not required that the demand profile $\boldsymbol{t}$ be infeasible. If $\boldsymbol{t}$ is feasible, the minimax solution $\boldsymbol{p}^*$ leads to spreading gains that satisfy $\sigma_i(\boldsymbol{p}^*) \le t_i$ for all $i = 1, \ldots, n$. Moreover, for any loss function of the form $\max_i l(\sigma_i(\boldsymbol{p}) - t_i)$ with a strictly increasing function l, the minimax solution is given by $\boldsymbol{p}^*$ from Theorem 4.

VII. Conclusion

We have defined the capacity region of a CDMA cellular network as the set of data rates combined with quality-of-transmission demands of n users that can be served by choosing an appropriate power setting below a maximum threshold. In this work, we show that the boundary of the capacity region with one mobile's power fixed and the rest unbounded is a shift of the boundary of some capacity region with changed parameters, but unlimited power. Using supporting hyperplanes and topological arguments, the convexity of the capacity region with limited power follows.

The boundary points of the capacity region are of particular interest since they represent system states of extreme load. In such states, the system is unable to provide more capacity to any of the users without drawing off capacity from others. Based on the above structural insights, we have explicitly determined minimax and Bayes power assignment strategies to balance conflicting interests of users in a rational way.

Interpreting the effective spreading gain as user payoff leads to a game theoretic setup where the capacity region plays the role of the payoff set. Future investigations will be devoted to access control strategies by using appropriate utility functions in this framework.

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