Optimal Rate and Power Allocation for OFDM in the Presence of Channel Uncertainty

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***Abstract*—Adaptive modulation of subcarriers for orthogonal frequency division multiplexing (OFDM) has been shown to improve system performance significantly. As a prerequisite, accurate channel state information (CSI) must be available at the transmitter. For time varying, noisy channels, however, this is difficult to achieve in practical systems. In this paper, we consider optimal subcarrier assignment under incomplete CSI. Soft estimates and the Cramer-Rao bound are used to analyze the effect of CSI errors. As an important control parameter, the frame length is coming in. We investigate analytically its influence on the performance of adaptive modulation in OFDM systems for the rate and power optimization problems. Simulations demonstrate that the frame length may not be neglected as an important overall system performance parameter.**

I. Introduction

OFDM forms a method of low-complexity to combat the effects of delay spread and frequency selective fading for high-speed wireless data transmission. It is presently used in many digital communication systems, such as digital audio and video broadcasting (DAB/DVB), wireless local area networks (WLAN) and worldwide interoperability for microwave access (WiMAX). In OFDM, the transmission band is divided into orthogonal subcarriers. If the bandwidth of subcarriers is sufficiently narrow, the subcarriers are subject to flat fading with different channel gains. A detailed description of this method is presented, e.g., in [1], [2].

To improve the bandwidth efficiency and enhance the system performance, different modulation schemes on different subcarriers may be employed in order to adapt to individual channel gains, a method also called bit loading, see [3]. In order to take advantage of adaptive modulation, accurate CSI is required at the transmitter. In practical wireless communications, however, due to noisy and time varying channels only imperfect CSI is available. The induced performance degradation is studied in [4], [5].

In this paper, we consider soft channel estimation, providing additional information on the channel uncertainty, cf. [6], and quantify the performance degradation of adaptive modulation for OFDM. By considering the Cramer-Rao lower bound (CRLB) on the error variance we derive universal results not depending on a specific estimation method.

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The paper is organized as follows. Section II presents the system model and assumptions. In Section III, the channel estimation error is characterized and its variance is lower bounded by the CRLB. Furthermore, its impact on data detection is analyzed. Based on this analysis, we study the effect of the channel uncertainty information on adaptive modulation for both the margin- and rate-adaptive problems in Section IV. Numerical results are given in Section V, which indicate that the frame length is an important control parameter for system performance.

II. System Model And Assumptions

Consider an OFDM system with N subcarriers. The data stream is divided into frames. Each frame consists of K OFDM symbols and may include pilot symbols or not. It is assumed that transmissions are subject to frequency selective fading and that the channel is invariant within a frame. After performing synchronization, removing cyclic prefix and executing fast Fourier transform (FFT) at the receiver, a received OFDM symbol may be written as

$$\mathbf{y} = \mathbf{D_h x} + \boldsymbol{\omega}, \tag{1}$$

where the $N \times 1$ vectors $\mathbf{y}$ and $\mathbf{x}$ refer to the received and transmitted OFDM symbols, respectively. The additive complex Gaussian noise vector is denoted by $\boldsymbol{\omega}$ with distribution $\boldsymbol{\omega} \sim \mathcal{CN}(\mathbf{0}, \sigma_\omega^2 \mathbf{I_N})$. Diagonal matrix $\mathbf{D_h} = diag(h[1], \ldots, h[N])$ contains perfect CSI, where $h[n]$ is the channel coefficient of the nth subcarrier in frequency domain.

In practice, reliable transmission always requires very low bit-error rates (BERs), like quasi-error-free in DVB, which means less than one error event per hour, corresponding to BER $= 10^{-10}$ to 10^{-11}, see [7]. Therefore, we can assume that our systems throughout this paper are operated in the high signal-to-noise ratio (SNR) range, where low BERs may be guaranteed.

III. Imperfect CSI

Due to noisy channel estimation and unavoidable delay between the transmitter and the receiver, it is impossible to obtain perfect CSI in practice. However, we may neglect the delay by assuming that the channel remains invariant for a sufficiently long period of time. Then we can focus on the effect of imperfect CSI only subject to channel estimation.

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A. Channel Estimation Error

The channel estimation error can be measured by the average mean square error (MSE) over subcarriers, which is defined as

$$\text{MSE} = E\Big\{\frac{1}{N}\sum_{n=1}^{N}|h[n]-\hat{h}[n]|^2\Big\},$$

where $\hat{h}[n]$ denotes the estimated channel coefficient of the nth subcarrier in frequency domain. If the multipath components experience independent Rayleigh fading, it follows that all subcarriers undergo identical Rayleigh fading, then the MSE may be written as

$$\text{MSE} = E\big\{|h[n]-\hat{h}[n]|^2\big\}.$$

For the sake of simplicity, we focus on one of the subcarriers and suppress the subcarrier index within this section. Then (1) simplifies to

$$y = hx + \omega. \tag{2}$$

With the least-squares channel estimation, the estimated channel coefficient is denoted by $\hat{h}$, derived as

$$\hat{h} = \frac{y}{x} = h + \underbrace{\frac{\omega}{x}}_{e}.$$

Obviously, h may be written as

$$h = \hat{h} - e.$$

If x is known or correctly detected by the receiver, the channel estimation error e is an independent zero-mean complex Gaussian random variable with variance $\sigma_e^2 = \sigma_\omega^2/|x|^2$, which can be interpreted as the channel uncertainty of h. Therefore, the estimated channel coefficient $\hat{h}$ can be treated as the expectation of h. Then h can be viewed as an zero-mean complex Gaussian random variable with variance σ_e^2. This is used as an assumption in [8], [9]. From this point of view it holds that

$$\text{MSE} = \sigma_e^2. \tag{3}$$

Hence, σ_e^2 may be also interpreted as reliability information on $\hat{h}$ [6]. The pair $(\hat{h}, \sigma_e^2)$ is called a soft channel estimate, it extends the channel estimate $\hat{h}$ by the channel uncertainty σ_e^2.

B. Effective Noise

Based on the decomposition above, (2) may be written as

$$y = \hat{h}x \underbrace{-ex + \omega}_{\eta}.$$

The latter term $\eta = -ex + \omega$ is called effective noise. It comprises the channel noise and the channel estimation error.

Since e is assumed stochastically independent with zero mean over OFDM symbols, it easily follows that

$$\sigma_\eta^2 = E(|-ex+\omega|^2) = P\sigma_e^2 + \sigma_\omega^2, \tag{4}$$

where $P = E\{|x|^2\}$ is the transmit power of data symbols.

The likelihood function with soft information included is derived as

$$\begin{aligned} p(y|x) &= \int_{-\infty}^{\infty} p(y|x,h)p(h)\mathrm{d}h \\ &= \frac{1}{\pi(|x|^2\sigma_e^2+\sigma_\omega^2)}\exp\left(-\frac{|y-\hat{h}x|^2}{|x|^2\sigma_e^2+\sigma_\omega^2}\right), \end{aligned} \tag{5}$$

where

$$p(y|x,h) = \frac{1}{\pi\sigma_\omega^2}\exp\left(-\frac{|y-hx|^2}{\sigma_\omega^2}\right)$$

is used for the conventional data detection with perfect CSI. Based on (5) the maximum-likelihood detection rule with imperfect CSI becomes

$$\hat{x} = \max_{\tilde{x}}\ p(y|\tilde{x}),$$

where $\tilde{x}$ ranges over the space of all possible data symbols.

C. The CRLB for the MSE of Channel Estimation

Channel estimation in OFDM systems has drawn a lot of attention over the last years. There are essentially two types of methods: pilot-based channel estimation and joint channel estimation and data detection.

The first approach is based only on pilot symbols, as depicted by Fig. 1.

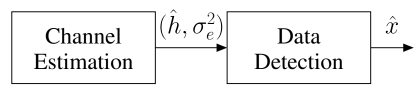

Fig. 1. Pilot-based channel estimation.

Its MSE can be expressed by the CRLB. In [10], the CRLB of the pilot-based channel estimation is given as

$$\overline{\text{MSE}} = \frac{a}{K_t SNR}, \tag{6}$$

where $a = L/N$ is determined by the channel characteristics and N, and L is the length of channel impulse response [11]. There are K_t pilot OFDM symbols in each frame. Further, $\overline{\text{MSE}}$ is normalized by the expected channel gain $E\{|h|^2\}$.

The second method additionally utilizes data symbols. Channel estimation and data detection are jointly performed, mutually benefiting from each other. To reduce complexity, joint channel estimation and data detection is performed iteratively as depicted in Fig. 2, cf. [11], [12], [13].

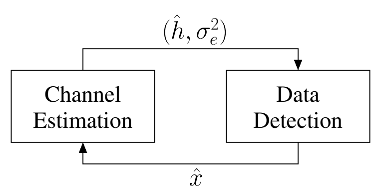

Fig. 2. Joint channel estimation and data detection

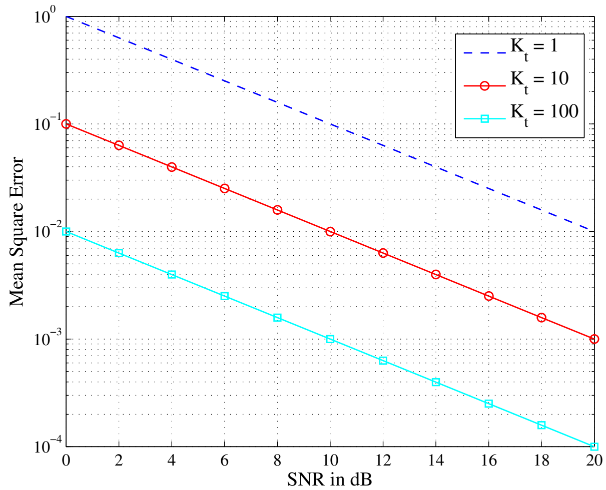

Fig. 3. The Cramer-Rao lower bound of the mean square error of channel estimation for $K_t = 1$, $K_t = 10$ and $K_t = 100$ with a = 1.

Under the assumption of high SNR with low BERs, the MSE of joint channel estimation and data detection is close to the CRLB. The CRLB at high SNR can be understood intuitively. It is given by the MSE of the least-squares channel estimation based on full knowledge of all data symbols. In this case, K_t in (6) is equal to the frame length K. In Fig. 3 the CRLB is plotted against the SNR for different values of K_t.

By combining (3) and (6) the variance of the channel estimation error is obtained as

$$\sigma_e^2 = \frac{a\sigma_\omega^2}{P_t K_t}, \tag{7}$$

where P_t denotes the transmit power of pilot symbols. If the AWGN noise power is assumed constant for each user, the channel estimation error is affected by the transmit power of pilot symbols and the frame length. However, if (7) is inserted in (4), then

$$\sigma_\eta^2 = \sigma_\omega^2\Big(1 + \frac{aP}{P_t K_t}\Big), \tag{8}$$

which shows that the overall noise consisting of the AWGN and the channel estimation error depends on the transmit power of data symbols.

If the pilot-based channel estimation is applied and the same transmit power is allocated to the data and pilot symbols, or if the joint channel estimation and data detection is used, we have the case $P_t = P$. Then (8) may be simplified to

$$\sigma_\eta^2 = \sigma_\omega^2\Big(1 + \frac{a}{K_t}\Big), \tag{9}$$

which will be used in the following. Note that (9) has only one variable K_t, the number of OFDM symbols used for channel estimation in each frame, while the additive noise power is fixed.

IV. Adaptive Modulation With Imperfect CSI

With the results from the previous section the received OFDM symbol (1) reads as

$$\mathbf{y} = \mathbf{D}_{\hat{\mathbf{h}}}\mathbf{x} + \boldsymbol{\eta}, \tag{10}$$

where $\mathbf{D}_{\hat{\mathbf{h}}} = diag\{\hat{h}[1], \ldots, \hat{h}[N]\}$ is the estimate of $\mathbf{D}_{\mathbf{h}}$. The noise $\boldsymbol{\eta}$ is a zero-mean complex Gaussian random vector with distribution $\boldsymbol{\eta} \sim \mathcal{CN}(\mathbf{0}, \sigma_\eta^2\mathbf{I_N})$.

In adaptive OFDM systems, different modulation schemes are used for subcarriers such that varying channel qualities are considered. However, only imperfect CSI is available at the transmitter. Here, the effective noise power (9) is used to include the channel estimation error.

Resource allocation problems can be generally divided into two groups according to different constraints, see [1]. One criterion is to maximize the performance margin while satisfying data transmission requirements and is called the margin-adaptive (MA) problem. On the other hand, the number of bits per OFDM symbol may be maximized subject to a fixed total transmit power constraint. This approach is called the rate-adaptive (RA) problem.

A. The Margin-Adaptive Problem

The margin-adaptive problem aims to minimize the total transmit power while satisfying the data rate and BER requirements. Considering the effective noise, which is affected by the additive noise power and especially by K_t, the margin-adaptive problem reads as

$$\min \sum_{n=1}^{N} P[n] \tag{11}$$

subject to :

$$\begin{aligned}
C1:&\quad \sum_{n=1}^{N} r[n] \geq R\\
C2:&\quad r[n] \leq M, \qquad \forall n \in \{1, \ldots, N\}\\
C3:&\quad r[n] = \log_2\left(1 + \frac{P[n]|\hat{h}[n]|^2}{\Gamma\sigma_\omega^2(1+\frac{a}{K_t})}\right),
\end{aligned}$$

where R in $C1$ denotes the minimal required data rate and M in $C2$ denotes the maximal number of bits that can be transmitted over each subcarrier in one OFDM symbol. The power-rate function $C3$ includes the estimation error as given in (10). The SNR gap Γ is determined by the required BER. For example, the SNR gap is 8.8 dB for uncoded square quadrature amplitude modulation (QAM) and BER $= 10^{-6}$.

By assuming *perfect* channel knowledge, the problem of optimal rate allocation is solved by water-filling, see [1], [14],

$$r[n] = \left[\log_2\left(\frac{\lambda|h[n]|^2}{\ln(2)\Gamma\sigma_w^2}\right)\right]^+, \tag{12}$$

where $[x]^+ = \max(x, 0)$ and λ is the water-level, determined by the rate constraint.

Algorithm 1 Water-Filling

Require: $\min \sum_{n\in\mathcal{D}} P[n]$
Ensure: $\sum_{n\in\mathcal{D}} r[n] = R,\ r[n] \geq 0$ and $r[n] \leq M$
  initialization
    $\mathcal{D} \leftarrow \{1, \ldots, N\}$
    $\mathcal{A} \leftarrow \emptyset$
  repeat
    $\lambda \leftarrow$ (14)
    $r[n] \leftarrow$(13), $n \in \mathcal{D}$
    $\mathcal{S} \leftarrow \{n \in \mathcal{D} \mid r[n] = 0\}$
    $\mathcal{L} \leftarrow \{n \in \mathcal{D} \mid r[n] > M\}$
    if $\mathcal{S} \neq \emptyset$ **then**
      $\mathcal{D} \leftarrow \mathcal{D} \setminus \mathcal{S}$
    else if $\mathcal{L} \neq \emptyset$ **then**
      $\mathcal{A} \leftarrow \mathcal{A} \cup \mathcal{L}$
      $\mathcal{D} \leftarrow \{1, \ldots, N\}/\mathcal{A}$
      $R \leftarrow R - M \times |\mathcal{L}|$
    end if
  until $\mathcal{S} = \emptyset$ and $\mathcal{L} = \emptyset$
  $\mathcal{D} \leftarrow \mathcal{A} \cup \mathcal{D}$
  output
    $r[n], P[n] \leftarrow$ the inverse of $C3, n \in \mathcal{D}$

By utilizing the *imperfect* channel knowledge given by an arbitrary channel estimator, the optimal solution to (11) can be derived as

$$r[n] = \left[\log_2\left(\frac{\lambda|\hat{h}[n]|^2}{\ln(2)\Gamma\sigma_w^2(1+\frac{a}{K_t})}\right)\right]^+. \quad (13)$$

Compared to (12), the rate allocation in (13) is more conservative because of the incomplete CSI. Convexity of (11) ensures that the optimal solution is achieved at equality in $C1$. The water-level is given by

$$\lambda = \ln(2)\Gamma\sigma_\omega^2(1+\frac{a}{K_t})2^{\frac{R}{d}}\left(\prod_{n\in\mathcal{D}}\frac{1}{|\hat{h}[n]|^2}\right)^{\frac{1}{d}}, \quad (14)$$

where $\mathcal{D}$ denotes the set of all used subcarriers with order d.

Algorithm 1 returns the solution to (11). Set $\mathcal{A}$ denotes the set of the subcarriers that achieve the maximal rate. First $\mathcal{D}$ is initialized as containing all subcarriers instead of only the subcarrier with the highest channel gain, since a high data rate is often required and most of the subcarriers are normally used in practice. In the loop above, the subcarriers loaded with more than M bits per OFDM symbol are moved from $\mathcal{D}$ to $\mathcal{A}$, provided there arise no negative rates for other subcarriers. Then $\mathcal{D}$ must be set to contain all subcarriers except those in $\mathcal{A}$ to investigate the possibility of using the before removed subcarriers.

B. The Rate-Adaptive Problem

In the rate-adaptive problem, the objective function is to maximize the overall rate. Furthermore, the rate constraint $C1$ in (11) is substituted by a power constraint. The resulting problem is concave. The rate-adaptive problem considering effective noise is solved similarly as the margin-adaptive one in the previous subsection, cf. [15]. The optimal power allocation with perfect CSI is given as

$$P[n] = \left[\frac{1}{\ln(2)\beta} - \frac{\Gamma\sigma_\omega^2}{|\hat{h}[n]|^2}\right]^+, \quad (15)$$

while the optimal solution for incomplete channel knowledge reads as

$$P[n] = \left[\frac{1}{\ln(2)\beta} - \frac{\Gamma\sigma_\omega^2(1+\frac{a}{K_t})}{|\hat{h}[n]|^2}\right]^+, \quad (16)$$

where β denotes the Lagrangian multiplier for RA and is determined by the power constraint. Compared to (15), (16) allocates less power on each subcarrier due to the impact of the channel estimation error.

V. Numerical Results

In this section, numerical results are presented in order to quantify the influence of the soft channel estimates on adaptive modulation in realistic environments. In the simulation, we use a frequency selective channel that is modeled as consisting of 16 independently Rayleigh distributed multipaths with an exponential attenuation profile. The expected channel gain on each subcarrier is normalized to one and the AWGN power is set to be -5 dB, which means $E\{|h[n]|^2\}/\sigma_\omega^2 = 5$ dB. The OFDM system has 16 subcarriers and the maximal rate on one subcarrier is $M = 6$. The constant a in (7) is set to one. The imperfect CSI is provided by the least-squares channel estimation. For an intended BER of 10^{-6}, Fig. 4 gives the total transmit power for MA and Fig. 5 gives the total achieved rate for RA with imperfect CSI in comparison with perfect CSI.

We can interpret the case $K_t = 1$ as an OFDM system using pilot-based channel estimation, where each frame includes one pilot symbol. The other cases of $K_t = 10$ and $K_t = 100$ can be viewed as OFDM systems using joint channel estimation and data detection with different frame lengths, which means $K_t = K$ here.

It can be seen that the increment of power consumption due to imperfect CSI in Fig. 4 and the decrement of rate achievement by using imperfect CSI in Fig. 5 becomes larger along the horizontal axis. This has mainly two reasons. One is the nonlinearity of the power-rate function. The other is that more additional power in MA or more compensating rate in RA is needed if more subcarriers are used. This happens when the required rate in MA or the power constraint in RA increases. It can be seen that the curves increase rapidly at large required rates in Fig. 4. Moreover, the curves grow only moderately at high constraint powers in Fig. 5. This behavior is caused by the maximal rate constraint per OFDM symbol on each subcarrier.

Moreover, on the vertical axis, compared to perfect CSI, the increment of total transmit power due to imperfect CSI becomes smaller in Fig. 4 (around 42% for $K_t = 1$, 5% for $K_t = 10$ and 0.58% for $K_t = 100$), and the total achieved

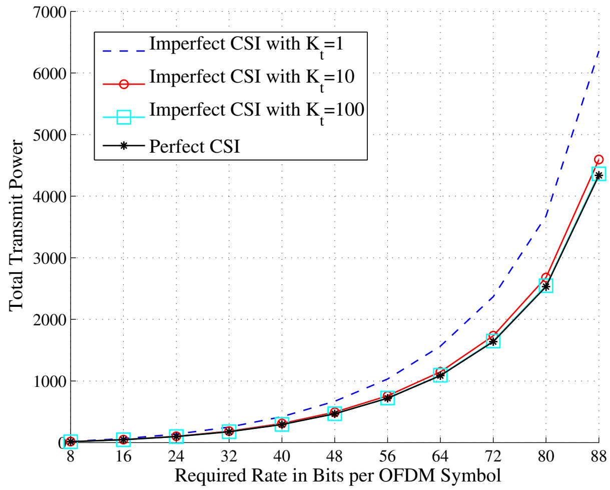

Fig. 4. MA: Total transmit power vs. required rates.

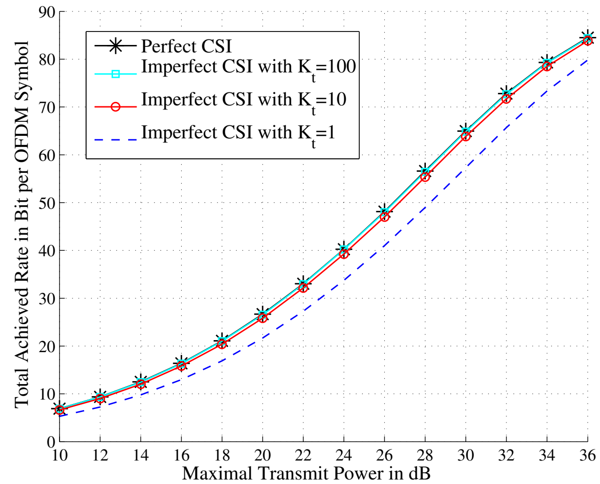

Fig. 5. RA: Total achieved rates vs. power constraints.

rate with imperfect CSI approaches to the one with perfect CSI in Fig. 5, when K_t becomes larger, which means that an increasing number of pilot or data symbols are used for channel estimation. In other words, the additional power in MA and compensating rate in RA for the channel estimation error tends to zero as K_t tends to infinity, since

$$\lim_{K_t \to \infty} \overline{\mathrm{MSE}} = \lim_{K_t \to \infty} \frac{a\sigma_\omega^2}{P_t K_t} = 0.$$

However, if pilot-based channel estimation is used, a large number of pilot symbols per frame would deteriorate the bandwidth efficiency. For joint channel estimation and data detection, K cannot be set very large either, since the channel stability within a sufficiently long period of time cannot be guaranteed. Moreover, the unavoidable delay cannot be neglected. For example, for downlink transmission in WiMAX each frame contains 24 to 36 OFDM symbols, see [16]. For these reasons, the number of symbols aiding channel estimation in a frame becomes considerably large.

VI. Conclusion

In this paper, we have studied noisy channel estimation in OFDM. As an approximation, we used the CRLB as channel estimation error. With this approach, we found that the effective noise power depends only on the number of OFDM symbols used for channel estimation per frame. The sheer additive noise power has been replaced by the extended noise power in the power-rate function. Under this replacement, our numerical results have shown that the performance of the resource allocation with imperfect CSI converges to the one with perfect CSI when the number of OFDM symbols used for channel estimation in a frame increases.

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