

Maximum Entropy Rate of Markov Sources for Systems With Non-regular Constraints

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Abstract

Using the concept of discrete noiseless channels, it was shown by Shannon in *A Mathematical Theory of Communication* that the ultimate performance of an encoder for a constrained system is limited by the combinatorial capacity of the system if the constraints define a regular language. In the present work, it is shown that this is not an inherent property of regularity but holds in general. To show this, constrained systems are described by generating functions and random walks on trees.

1. INTRODUCTION

A constrained system allows the transmission of input sequences of weighted symbols that fulfill certain constraints on the symbol constellations. Constrained systems have been of recent interest, e.g., in the context of storage systems [1]. A natural question is how to efficiently encode a random source such that it becomes a valid input for a constrained system [2]. Furthermore, it is of interest to determine the ultimate performance of such an encoder. This leads to the notion of the capacity of constrained systems.

Previous work: Shannon [3] investigated the capacity of constrained systems within the framework of the *discrete noiseless channel* (DNC). For the case where the constraints form a regular language [4], it was stated in [3, Theorem 8] that the maximum entropy rate R of a valid input process is equal to the *combinatorial capacity* C , which is defined as

$$C = \limsup_{\nu \rightarrow \infty} \frac{\ln N(\nu)}{\nu} \quad (1)$$

where ν denotes the length of the sequences and $N(\nu)$ denotes the number of distinct sequences of length ν that are accepted by the considered DNC. Here and

hereafter, \ln denotes the natural logarithm. A detailed proof of the equality between R and C was recently given in [5]. This proof is heavily based on the regularity of the constraints. However, it is not clear whether this equality is an inherent property of regular languages or whether it holds in general. It should be noted that sequences with non-regular constraints have been of research interest recently, e.g., in [6]. An early treatment of DNCs can be found in [7].

Contributions: In this paper, we use the framework of *general DNCs* as introduced in [8] to show the following. If the set of valid input sequences for a constrained system can be generated by a Markov process, then the maximum entropy rate of such a process is given by the combinatorial capacity of the system, irrespective of whether the constraints are regular or not. Our result can be seen as a generalization of Shannon's result [3, Theorem 8] to general DNCs and in particular non-regular DNCs. Furthermore, since our derivations also apply for the regular case, they also serve as a new way to derive [3, Theorem 8].

The remainder of the paper is organized as follows. In Section 2, we present the framework of general DNCs and the calculation of combinatorial capacities by generating functions as introduced in [8]. We then define in Section 3 Markovian input processes and entropy rates for general DNCs. In Section 4, we define the maximum entropy rate R of general DNCs and for sake of illustration, we show for two simple examples that R is equal to the combinatorial capacity C . Finally, in Section 5 we prove that $R = C$ holds for general DNCs.

2. DISCRETE NOISELESS CHANNELS

To calculate the combinatorial capacity of general DNCs, we interpret generating functions as functions on the complex plane and investigate their convergence behavior. This approach, mostly referred to as *analytic*

combinatorics, is discussed in detail in [9]. We consider a more general case since we allow non-integer valued symbol weights. In order to handle this situation, we use general Dirichlet's series [10] instead of Taylor series as generating functions.

2.1. Definitions and Notation

Our definition of DNCs as presented next mainly follows the one given in [8].

Definition 1. A DNC $\mathcal{A} = (A, \omega)$ consists of a countable set A of strings accepted by the channel and an associated weight function $\omega: A \rightarrow \mathbb{R}^{\oplus}$ (\mathbb{R}^{\oplus} denotes the nonnegative real numbers) with the following property. If $a, b \in A$ and $ab \in A$ (ab denotes the concatenation of a and b), then $\omega(ab) = \omega(a) + \omega(b)$. By convention, the empty string ε is always an element of A and the weight of ε is equal to zero, i.e., $w(\varepsilon) = 0$.

Definition 2. Let $\mathcal{A} = (A, \omega)$ represent a DNC. We define the *generating function* of \mathcal{A} by

$$G_{\mathcal{A}}(s) = \sum_{a \in A} e^{-\omega(a)s}, \quad s \in \mathbb{C} \quad (2)$$

where \mathbb{C} denotes the set of complex numbers.

Let Ω denote the set of distinct string weights of elements in A . We order and index the set Ω such that $\Omega = \{\nu_k\}_{k=1}^{\infty}$ with $\nu_1 < \nu_2 < \dots$. For every $\nu_k \in \Omega$, $N(\nu_k)$ denotes the number of distinct strings of weight ν_k that are accepted by the channel. We can now write the generating function as

$$G_{\mathcal{A}}(s) = \sum_{k=1}^{\infty} N(\nu_k) e^{-\nu_k s}. \quad (3)$$

Since the coefficients $N(\nu_k)$ result from an enumeration, they are all nonnegative. The combinatorial capacity of a DNC as defined in (1) can now be written as

$$C = \limsup_{k \rightarrow \infty} \frac{\ln N(\nu_k)}{\nu_k}. \quad (4)$$

2.2. DNCs of Interest

Throughout this paper, we restrict our attention to DNCs where the ordered set of string weights $\{\nu_k\}_{k=1}^{\infty}$ is *not too dense*, that is, there exists some constant $L \geq 0$ and some constant $K \geq 0$ such that for any integer $n \geq 0$

$$\max_{\nu_k < n} k \leq Ln^K. \quad (5)$$

Otherwise, the number of possible string weights in the interval $[n, n+1]$ increases exponentially with n , in which case the definition of combinatorial capacity given in (4) is not appropriate. This is illustrated in the following example.

Example 1. Let $N(\nu_k)$ denote the coefficients of the generating function of some DNC. Assume $N(\nu_k) = 1$ for all $k \in \mathbb{N}$ and assume further

$$\max_{\nu_k < n} k = \lceil R^n \rceil \quad (6)$$

for some $R > 1$. According to (4), the capacity of the DNC is then equal to zero because of $\ln N(\nu_k) = 0$ for all $k \in \mathbb{N}$. However, the channel accepts R^n distinct strings of weight smaller than n . The average amount of data per string weight that we can transmit over the channel is thus lower-bounded by $\ln R^n / n = \ln R$, which is according to the assumption greater than zero. \triangleleft

For a DNC $\mathcal{A} = (A, \omega)$ where A is generated over a finite set of symbols, the restriction (5) is automatically fulfilled [5, Appendix A], implying that virtually any constrained system of practical interest fulfills (5). Not too dense sequences have another interesting property, which we will need in our later derivations. We state it in the following lemma.

Lemma 1. *If a series $\{a_k\}_{k=1}^{\infty}$ is not too dense and if $0 \leq x < 1$, then the series $\sum_{k=1}^{\infty} x^{a_k}$ converges.*

See [5, Appendix A] for a proof of this lemma.

2.3. Calculating the Capacity

For a DNC of interest, we want to calculate the combinatorial capacity as given in (4). An explicit formula for regular DNCs was provided in [3, Theorem 1]. A detailed derivation of this formula for DNCs with regular constraints and non-integer valued symbol weights can be found in [5]. In [8], it was shown that the combinatorial capacity (4) is determined by the region of convergence (r.o.c.) of the corresponding generating function for any DNC with the set of possible string weights $\{\nu_k\}_{k=1}^{\infty}$ being not too dense. We restate this theorem here.

Theorem 1. *Let $\mathcal{A} = (A, \omega)$ be a DNC with the generating function $G_{\mathcal{A}}(s)$. The combinatorial capacity C of \mathcal{A} is given by $C = Q$ where $\Re\{s\} > Q$ ($\Re\{s\}$ denotes the real part of s) is the r.o.c. of $G_{\mathcal{A}}(s)$, that is,*

$$C = \limsup_{k \rightarrow \infty} \frac{\ln N(\nu_k)}{\nu_k} = Q. \quad (7)$$

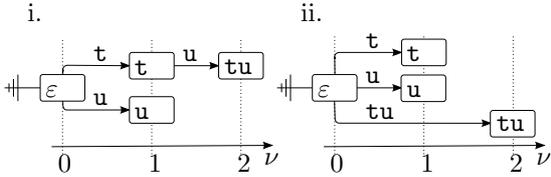


Figure 1: Two different representations of the DNC $\mathcal{A} = (A, w)$ by a tree. The DNC \mathcal{A} is given by $A = \{\varepsilon, \mathfrak{t}, \mathfrak{u}, \mathfrak{tu}\}$ with $w(\mathfrak{t}) = w(\mathfrak{u}) = 1$.

Theorem 1 applies for general DNCs with possibly non-integer valued symbol weights and arbitrary constraints on the symbol constellations. It can be interpreted as the general form of the *Exponential Growth Formula*. In [9, Theorem IV.7], the Exponential Growth Formula was stated for DNCs with integer valued weights and arbitrary constraints. The latter version of the Exponential Growth Formula was used in [6] to calculate the combinatorial capacity (4) of a non-regular DNC with integer valued symbol weights.

3. INPUT SOURCES FOR DNCs

The purpose of this section is to define Markovian input processes and the corresponding entropy rates for general DNCs. First, we represent the set of strings that are accepted by a general DNC by a tree and second, we define a Markovian input process as a walk on this tree and give a formula for its entropy rate. We postpone the problem of finding the maximum entropy rate to the next section.

3.1. Representing DNCs by Trees

We represent a DNC $\mathcal{A} = (A, \omega)$ by a tree $T_{\mathcal{A}}$ consisting of a root, labelled and weighted branches, and paths resulting from the concatenation of branches. We restrict our considerations to paths that start at the root. For each such path, we display its label at the corresponding end node. We do not allow distinct paths to have the same label. A DNC \mathcal{A} is represented by a tree $T_{\mathcal{A}}$ if there is a one-to-one mapping from A to the path labels. Note that only the *set of paths* in $T_{\mathcal{A}}$ is uniquely determined by this mapping, but not how these paths are formed by branches. See Figure 1 for an example of this ambiguity. In this figure, a branch is represented by an arrow, its weight by the distance between start and end node, and its label is written above the arrow. Notice that the set of paths represented by the node labels displayed in the rectangles is the same for the tree in Figure 1 i and the tree in Figure 1 ii. The DNC has a finite set A of accepted sequences, therefore, the tree

representations are finite. However, DNCs of non-zero combinatorial capacity have infinite sets of accepted strings and as a consequence also infinite tree representations. Surprisingly, we will see in the following that although the tree representation of a DNC is not unique, as long as it allows the definition of a Markov input source, the maximum entropy rate of this source will not depend on the chosen tree representation.

3.2. Markovian Input Sources

For a DNC $\mathcal{A} = (A, \omega)$, we assume that every branch in the tree representation $T_{\mathcal{A}}$ has subsequent branches. We can then define an input source by a Markov process $X = \{X_l\}_{l=1}^{\infty}$, where X_l chooses randomly among the branches that start at the end node of the realization of X_{l-1} . Every realization of $\mathbf{X}^{(l)} = (X_1, \dots, X_l)$ is thus a path in $T_{\mathcal{A}}$ starting at the root and consisting of l branches. The support of $\mathbf{X}^{(l)}$ is given by the set of all such paths $\mathbf{x}^{(l)}$ and we denote it by $\mathfrak{X}^{(l)}$. Note that for $\mathcal{A} = (A, \omega)$, we have

$$A = \bigcup_{l=1}^{\infty} \mathfrak{X}^{(l)}. \quad (8)$$

Whenever it follows directly from the context, we omit for simplicity the superscript l and write \mathbf{x} instead of $\mathbf{x}^{(l)}$. For all $\mathbf{x} \in \mathfrak{X}^{(l)}$, we have for the probability mass function (PMF) $p_{\mathbf{X}^{(l)}}$ of $\mathbf{X}^{(l)}$

$$p_{\mathbf{X}^{(l)}}(\mathbf{x}) = \mathbb{P}[X_1 = x_1] \prod_{i=2}^l \mathbb{P}[X_i = x_i | X_{i-1} = x_{i-1}]. \quad (9)$$

We conclude that the existence of a tree representation $T_{\mathcal{A}}$ where each branch has subsequent branches is equivalent to the existence of a Markovian input source for \mathcal{A} . Note that Regular DNCs can be represented by *finite state machines* (FSMs) [4] and the tree representation can be obtained from the corresponding FSM. The resulting tree representation then has automatically the property that each branch has subsequent branches.

Following [3],[5], the *entropy rate* \bar{H} of X is given by

$$\bar{H}(X) = \limsup_{l \rightarrow \infty} \frac{H(\mathbf{X}^{(l)})}{L_l} \quad (10)$$

where L_l is equal to the average weight of all $\mathbf{x} \in \mathfrak{X}^{(l)}$ with respect to (w.r.t.) the PMF of $\mathbf{X}^{(l)}$ and where $H(\mathbf{X}^{(l)})$ denotes the entropy of $\mathbf{X}^{(l)}$ in nats.

4. PROBLEM STATEMENT

We now come to the key topic of this paper: the maximization of the entropy rate of input processes for general DNCs.

4.1. Maximum Entropy Rate

Definition 3. We define the *maximum entropy rate* R of a DNC by

$$R = \max_X \bar{H}(X). \quad (11)$$

where the maximum is taken over all Markovian processes X that generate valid input sequences for the DNC.

Note that in [5], the term *probabilistic capacity* was used instead of maximum entropy rate. However, we prefer the latter term.

The entropy rate $\bar{H}(X)$ is maximized, if each term of the sequence on the right hand side of (10) is maximized. For each l , the maximum entropy per average branch weight

$$R_l = \max_{p_{\mathbf{X}^{(l)}}} \frac{H(\mathbf{X}^{(l)})}{L_l} \quad (12)$$

is given by the greatest positive real solution of the equation

$$\sum_{\mathbf{x} \in \mathfrak{X}^{(l)}} e^{-\omega(\mathbf{x})s} = 1. \quad (13)$$

In addition, for all $\mathbf{x} \in \mathfrak{X}^{(l)}$, the PMF of $\mathbf{X}^{(l)}$ that achieves this rate is uniquely given by

$$q_{\mathbf{X}^{(l)}}(\mathbf{x}) = e^{-\omega(\mathbf{x})R_l}. \quad (14)$$

These two properties of R_l were derived by using Lagrange Multipliers in [11] and they were independently derived in [12] by using the bound $\ln x \leq x - 1$. We offer an alternative proof by applying the information inequality [13], which states for the Kullback Leibler Distance $D(\cdot \|\cdot)$ of two PMFs p and q that

$$D(p \|\ q) \geq 0 \quad (15)$$

with equality if and only if $p = q$. We thus have

$$0 \geq -D(p_{\mathbf{X}^{(l)}} \|\ q_{\mathbf{X}^{(l)}}) \quad (16)$$

$$= \sum_{\mathbf{x} \in \mathfrak{X}^{(l)}} p_{\mathbf{X}^{(l)}}(\mathbf{x}) \ln \frac{q_{\mathbf{X}^{(l)}}(\mathbf{x})}{p_{\mathbf{X}^{(l)}}(\mathbf{x})} \quad (17)$$

$$= H(\mathbf{X}^{(l)}) - R_l L_l \quad (18)$$

which implies

$$\frac{H(\mathbf{X}^{(l)})}{L_l} \leq R_l \quad (19)$$

with equality if and only if $p_{\mathbf{X}^{(l)}} = q_{\mathbf{X}^{(l)}}$. Combining (10), (11), and (12), we have

$$R = \limsup_{l \rightarrow \infty} R_l = \limsup_{l \rightarrow \infty} \max_{p_{\mathbf{X}^{(l)}}} \frac{H(\mathbf{X}^{(l)})}{L_l}. \quad (20)$$

The form on the right hand side of (20) allows us to compare the maximum entropy rate of a DNC to its combinatorial capacity as given in (4). We illustrate this in the following by two simple examples.

Example 2. Let $\mathcal{A} = (A, \omega)$ represent a DNC that accepts all binary input sequences. The set A is thus given by $A = \{0, 1\}^*$ where $*$ denotes the regular operation *star* [4]. We assume the symbol weights $\omega(0) = \omega(1) = 1$. The combinatorial capacity is given by

$$C = \limsup_{k \rightarrow \infty} \frac{\ln N(\nu_k)}{\nu_k} \quad (21)$$

$$= \limsup_{k \rightarrow \infty} \frac{\ln 2^k}{k}. \quad (22)$$

To calculate the maximum entropy rate of \mathcal{A} , we note that for each $\mathbf{x} \in \mathfrak{X}^{(l)}$, we have $\omega(\mathbf{x}) = l$ and in addition, the cardinality of $\mathfrak{X}^{(l)}$ is given by $|\mathfrak{X}^{(l)}| = 2^l$. The average weight L_l of $\mathbf{x}^{(l)}$ is thus given by $L_l = l$ and maximizing the entropy rate reduces to maximizing the entropy of $\mathbf{X}^{(l)}$. The maximum entropy of $\mathbf{X}^{(l)}$ is given by $\max_{p_{\mathbf{X}^{(l)}}} H(\mathbf{X}^{(l)}) = \ln |\mathfrak{X}^{(l)}|$, see [13]. All together we have

$$R = \limsup_{l \rightarrow \infty} \max_{p_{\mathbf{X}^{(l)}}} \frac{H(\mathbf{X}^{(l)})}{L_l} \quad (23)$$

$$= \limsup_{l \rightarrow \infty} \frac{\max_{p_{\mathbf{X}^{(l)}}} H(\mathbf{X}^{(l)})}{l} \quad (24)$$

$$= \limsup_{l \rightarrow \infty} \frac{\ln |\mathfrak{X}^{(l)}|}{l} \quad (25)$$

$$= \limsup_{l \rightarrow \infty} \frac{\ln 2^l}{l}. \quad (26)$$

We see from (22) and (26) that the maximum entropy rate of \mathcal{A} is equal to the combinatorial capacity, that is, $R = C$. \triangleleft

Example 3. As in Example 2, we consider a DNC $\mathcal{A} = (A, \omega)$ that accepts all binary input sequences. However, we assume the symbol weights $\omega(0) = 1$ and $\omega(1) = 2$. To show that $C = R$ also holds in this case, we have to explicitly calculate C and R . To show equality by comparison as we did by (22) and (26) in

the previous example is no longer possible. To calculate the combinatorial capacity, we write the generating function of \mathcal{A} as

$$G_{\mathcal{A}}(s) = \sum_{m=0}^{\infty} (e^{-1s} + e^{-2s})^m. \quad (27)$$

The series converges if $\Re\{e^{-1s} + e^{-2s}\} < 1$, therefore, the combinatorial capacity C is by Theorem 1 given by the smallest positive real solution of

$$e^{-1s} + e^{-2s} = 1. \quad (28)$$

Let Y denote a random variable with support $\{0, 1\}$, and the associated weights $\omega(0) = 1$ and $\omega(1) = 2$. In addition, let L denote the average weight of Y . The maximum entropy rate of \mathcal{A} can then be calculated as

$$R = \limsup_{l \rightarrow \infty} \max_{p_{\mathbf{X}^{(l)}}} \frac{H(\mathbf{X}^{(l)})}{L_l} \quad (29)$$

$$= \limsup_{l \rightarrow \infty} \max_{p_Y} \frac{lH(Y)}{lL} \quad (30)$$

$$= \max_{p_Y} \frac{H(Y)}{L}. \quad (31)$$

By (13), it follows from the last line that R is also given by (28), thus $R = C$. \triangleleft

5. MAIN RESULT

Based on the concepts introduced in the previous sections, we can now state our main result.

Theorem 2. *If the set of valid input sequences of a DNC $\mathcal{A} = (A, \omega)$ can be generated by a Markov process (or equivalently, if the DNC can be represented by a tree where each branch has a subsequent branch), then the maximum entropy rate R of \mathcal{A} is equal to its combinatorial capacity C , that is,*

$$\limsup_{k \rightarrow \infty} \frac{\ln N(\nu_k)}{\nu_k} = \limsup_{l \rightarrow \infty} \max_{p_{\mathbf{X}^{(l)}}} \frac{H(\mathbf{X}^{(l)})}{L_l}. \quad (32)$$

We will prove this equality in the following. Although equality was shown in [5] for regular DNCs, to the best of our knowledge nobody has addressed the non-regular case until now.

Proof of Theorem 2. To prove the theorem, we show that the region of convergence of the generating function $G_{\mathcal{A}}(s)$ is given by $\Re\{s\} > R$. The theorem then follows because of Theorem 1.

The maximum entropy rate R is given by (20), which is equivalent to the following. For every $\epsilon > 0$, it holds

that

$$R_l < R + \epsilon \quad \text{almost everywhere (a.e.)} \quad (33)$$

and

$$R_l > R - \epsilon \quad \text{infinitely often (i.o.)} \quad (34)$$

with respect to $l \in \mathbb{N}$ (the set of natural numbers). Since R_l is given by (13), this implies further

$$\sum_{\mathbf{x} \in \mathfrak{X}^{(l)}} e^{-\omega(\mathbf{x})[R+\epsilon]} < \sum_{\mathbf{x} \in \mathfrak{X}^{(l)}} e^{-\omega(\mathbf{x})R_l} = 1 \quad \text{a.e.} \quad (35)$$

and

$$\sum_{\mathbf{x} \in \mathfrak{X}^{(l)}} e^{-\omega(\mathbf{x})[R-\epsilon]} > \sum_{\mathbf{x} \in \mathfrak{X}^{(l)}} e^{-\omega(\mathbf{x})R_l} = 1 \quad \text{i.o.} \quad (36)$$

Because of (8), we can write the generating function as

$$G_{\mathcal{A}}(s) = \sum_{a \in A} e^{-\omega(a)s} \quad (37)$$

$$= \lim_{n \rightarrow \infty} \sum_{l=1}^n \sum_{\mathbf{x} \in \mathfrak{X}^{(l)}} e^{-\omega(\mathbf{x})s} \quad (38)$$

and we can use (35) and (36) to give bounds on $G_{\mathcal{A}}(s)$ around $s = R$. It follows directly from (36) that

$$\sum_{l=1}^n \sum_{\mathbf{x} \in \mathfrak{X}^{(l)}} e^{-\omega(\mathbf{x})[R-\epsilon]} \xrightarrow{n \rightarrow \infty} \infty. \quad (39)$$

For every $\epsilon > 0$, the generating function $G_{\mathcal{A}}(s)$ thus diverges for $\Re\{s\} \leq R - \epsilon$. It remains to show that it converges whenever $\Re\{s\} > R$. For some arbitrary but fixed $\epsilon_0 > 0$, define

$$D = \sum_{\{l \in \mathbb{N} \mid R + \epsilon_0 \leq R_l\}} \sum_{\mathbf{x} \in \mathfrak{X}^{(l)}} e^{-\omega(\mathbf{x})[R+\epsilon_0]} \quad (40)$$

Because of (33), the sum is taken over a finite number of terms, and as a result, D is a finite number. For every ϵ with $\epsilon_0 > \epsilon > 0$, we have

$$\sum_{l=1}^n \sum_{\mathbf{x} \in \mathfrak{X}^{(l)}} e^{-\omega(\mathbf{x})[R+2\epsilon]} = \sum_{l=1}^n \sum_{\mathbf{x} \in \mathfrak{X}^{(l)}} e^{-\omega(\mathbf{x})\epsilon} e^{-\omega(\mathbf{x})[R+\epsilon]} \quad (41)$$

$$\leq \sum_{l=1}^n \sum_{\mathbf{x} \in \mathfrak{X}^{(l)}} e^{-\nu_l \epsilon} e^{-\omega(\mathbf{x})[R+\epsilon]} \quad (42)$$

$$= \sum_{l=1}^n e^{-\nu_l \epsilon} \sum_{\mathbf{x} \in \mathfrak{X}^{(l)}} e^{-\omega(\mathbf{x})[R+\epsilon]} \quad (43)$$

$$\leq \sum_{l=1}^n e^{-\nu_l \epsilon} \sum_{\mathbf{x} \in \mathfrak{X}^{(l)}} e^{-\omega(\mathbf{x})R_l} + D \quad (44)$$

$$= \sum_{l=1}^n e^{-\nu_l \epsilon} + D. \quad (45)$$

The inequality in (42) holds because for every $l \in \mathbb{N}$, the weight of $\mathbf{x} \in \mathfrak{X}^{(l)}$ is lower bounded by $\omega(\mathbf{x}) \geq \nu_l$. We have inequality in (44), because of $\exp(-\nu_l \epsilon) < 1$ and (33). For those l for which $R_l \leq R + \epsilon$ does not apply, we add the correcting value D as defined in (40). We can now write the sum in (45) as

$$\sum_{l=1}^n e^{-\nu_l \epsilon} = \sum_{l=1}^n (e^{-\epsilon})^{\nu_l}. \quad (46)$$

For n tending to infinity, according to Lemma 1, this series converges, since $\{\nu_k\}_l^\infty$ is not too dense and since $\exp(-\epsilon) < 1$. We conclude that $G_{\mathcal{A}}(s)$ converges for $\Re\{s\} \geq R + 2\epsilon$.

If, for every $\epsilon > 0$, $G_{\mathcal{A}}(s)$ diverges for $\Re\{s\} \leq R - \epsilon$ and converges for $\Re\{s\} \geq R + \epsilon$, then the region of convergence of $G_{\mathcal{A}}(s)$ is given by $\Re\{s\} > R$. This concludes the proof of the theorem. \square

6. CONCLUSIONS

In this work, we showed that the equality of the combinatorial capacity and the maximum entropy rate of an input process holds for constrained systems in general and is not a consequence of regular constraints, which were considered in this context until now. In contrast to the proof of [3, Theorem 8] in [5] for the regular case, our proof for the general case is not constructive, so it remains a challenge to explicitly define capacity achieving input sources for constrained systems with non-regular constraints as the one considered in [6].

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