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Compressive Forwarding for Jointly Sparse Signals in Amplify-and-Forward Gaussian Relay Networks

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***Abstract*—This paper considers the problem of applying compressed sensing ideas to relay networks in order to increase the network throughput. We present a new method to perform forwarding which, when the signals transmitted by several sources are jointly sparse, enables to send a vector of dimension higher than the min-cut of the network. First, the jointly sparse source signals are mapped to a signal of lower dimension which can be transmitted over the network. Second, we allow source nodes to send at a so called *single-source min-cut rate* which is such that each node transmits without considering other sources transmitting simultaneously. Finally, sinks use compressed sensing in order to recover the transmitted sparse signal. Algebraically, the source signal is multiplied by a network matrix which needs to have the restricted isometry property to provide perfect reconstruction. We present algorithms to choose the network matrix coefficients as well as simulation results to show the validity of our approach.**

I. Introduction

Relay networks are models for communication systems where one or more sources transmit information to one or more sinks through relays. Fundamental results for a simple relay network with one source, one sink, and one relay were introduced in [1] and [2]. In [3], the authors present results for the Gaussian parallel network, where one source communicates to two relays which then forward information to a sink.

In our work [4] we have shown how Amplify-and-Forward Gaussian Relay Networks can be modeled by a graph and further by matrices. In general, the maximal dimension of the transmitted signal is bounded by the physical dimension of the network, i.e., the min-cut of the graph representing the network. In the present work we develop a framework to use compressed sensing methods in order to send sparse signals of high dimension over a network of low dimension. The signals at different sources do not need to be individually compressible. If the global source signal is composed of several jointly sparse signals located at several source nodes, our framework enables to perform distributed compressed sensing over the network.

Compressed sensing (see, e.g., in [5], [6], or [7]) is a new sampling method that enables to recover a sparse signal from much fewer measurements than the actual size of the signal (we call the size of a signal the dimension of the vector representing the signal). Let's assume one wants to sample a signal $\mathbf{z} \in \mathbb{C}^n$ which is k-sparse, i.e., it has no more than k nonzero components. Using a sampling matrix $\mathbf{\Phi} \in \mathbb{R}^{m \times n}$, the sampling result is $\mathbf{x} = \mathbf{\Phi z}$ with $\mathbf{x} \in \mathbb{C}^m$. To recover $\mathbf{z}$ from $\mathbf{x}$, one typically applies the following convex program

$$\begin{array}{ll} \text{minimize} & \|\hat{\mathbf{z}}\|_{l_1} \\ \text{subject to} & \mathbf{\Phi}\hat{\mathbf{z}} = \mathbf{x} \end{array} \quad (1)$$

where $\|.\|_{l_1}$ is the l_1 norm, i.e., $\|\hat{\mathbf{z}}\|_{l_1} = \sum_{i=1}^{i=n} |\hat{z}_i|$. If $\mathbf{\Phi}$ verifies certain conditions, detailed in Section II, then $\hat{\mathbf{z}}^* = \mathbf{z}$ where $\hat{\mathbf{z}}^*$ denotes the solution of the problem (1).

There exists already works using compressed sensing for communication systems. In [8] the authors present an approach to use compressed sensing, in a setup where many sensors transmit signals to a central node, exploiting the intra-correlation of the transmitted signals. In [9] the authors developed a general model for this problem, which exploits both inter- and intra-signal correlation. They consider some different sparsity models and developed recovery strategies as well as bounds on the required number of measurements to have perfect recovery. These works only considers single-hop networks. In contrast, our work focuses on multi-hop networks.

The central idea of the present paper is the following: a network can be represented by a matrix, if this matrix can be chosen such that it verifies the conditions that guarantee perfect recovery in the sense of compressed sensing, then this network might transport sparse signals of higher dimension than the physical size of the network.

The rest of the present paper is organized as follow. In Section II we introduce the basics of compressed sensing. In Section III we describe the network model supporting our results. In Section IV we introduce the central concept of compressive forwarding in relay networks. In Section V and Section VI we give algorithms to choose the network matrix for single-source and multi-source networks. In Section VII we present simulation results and in Section VIII we conclude this work.

II. Compressed Sensing Basics

A central question in compressed sensing is how to choose the sensing matrix $\mathbf{\Phi}$ to achieve perfect reconstruction of

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k-sparse signals and how small can m be (i.e., how few measurements are sufficient). In [6] the restricted isometry property (RIP) is introduced. It enables to measure if a given matrix will provide perfect reconstruction for given (n, m, k). A matrix $\mathbf{\Phi}$ satisfies the restricted isometry property of order k if there exists a constant $\delta_k < 1$, such that

$$(1-\delta_k)\|\mathbf{z}\|_{l_2}^2 \leq \|\mathbf{\Phi z}\|_{l_2}^2 \leq (1+\delta_k)\|\mathbf{z}\|_{l_2}^2 \quad (2)$$

holds for all k-sparse vectors $\mathbf{z}$. We note $\|.\|_{l_2}$ the Euclidean norm. In [10], it has been proven that if $\mathbf{\Phi}$ satisfies the RIP of order $2k$ with $\delta_{2k} < \sqrt{2}-1$, then the program (1) recovers all k-sparse vectors perfectly and makes a small error on non-k-sparse vectors. Matrices obeying the RIP are, for example, Gaussian matrices with $m \geq c\ k \log(n/k)$, where c is a constant and all entries are independent and identically distributed (i.i.d.) as $\mathcal{N}(0, 1/m)$.

It is however a hard problem to verify if the RIP holds. In [7] the authors introduce the nullspace property (NSP), which is a sufficient property for a sensing matrix to provide perfect reconstruction of k-sparse vectors. In contrast to the RIP, it is possible to verify the NSP using tractable methods described in [11]. We provide next the basics of the NSP. We define $g(\mathbf{x}, \mathbf{\Phi})$ as a function providing the solution of the program (1). A pair $(\mathbf{\Phi}, g)$ is said to be instance optimal of order k with constant C if

$$\|\mathbf{z} - g(\mathbf{x}, \mathbf{\Phi})\|_{l_1} \leq C\sigma_k(\mathbf{z}), \qquad \forall \mathbf{z} \in \mathbb{C}^n, \mathbf{x} = \mathbf{\Phi z}, \quad (3)$$

where $\sigma_k(\mathbf{z})$ is called the best k-term approximation of $\mathbf{z}$, i.e., the error made by approximating $\mathbf{z}$ by its k largest components only and setting all other components to zero. In other words, $(\mathbf{\Phi}, g)$ is instance optimal if performing compressed sensing causes 1) no errors when $\mathbf{z}$ is k-sparse and 2) an error proportional to the error of best k-term approximation when $\mathbf{z}$ is not k-sparse. A matrix $\mathbf{\Phi} \in \mathbb{R}^{m \times n}$ is said to have the NSP in l_1 of order k with constant C_k if and only if

$$\|\boldsymbol{n}\|_{l_1} \leq C_k \|\boldsymbol{n}_{\mathcal{T}^\mathcal{C}}\|_{l_1} \quad (4)$$

holds for all vectors $\boldsymbol{n}$ in the nullspace of $\mathbf{\Phi}$ and all sets of indices $\mathcal{T}$ of $\boldsymbol{n}$ with $\#\mathcal{T} \leq k$. The set $\mathcal{T}^\mathcal{C}$ denotes the complement of $\mathcal{T}$ and $\boldsymbol{n}_{\mathcal{T}^\mathcal{C}}$ is the restriction of $\boldsymbol{n}$ to $\mathcal{T}^\mathcal{C}$, i.e., $(n_{\mathcal{T}^\mathcal{C}})_i = n_i$ on $\mathcal{T}^\mathcal{C}$ and zero otherwise. If $\mathbf{\Phi}$ has the NSP of order k with constant $C_k > 0$, there exists a decoder that guarantees perfect recovery for a sparsity $k/2$. If $C_k < 2$ then the program (1) satisfies instance optimality of order k with a constant C such that $C = 2C_k/(2-C_k)$. It has been shown in [11] that the NSP can be verified in polynomial time using semidefinite programming. We will use the NSP in this paper to evaluate if a matrix is appropriate for compressed sensing.

III. Network Model

We consider networks that can be represented by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$. We call e the cardinality of $\mathcal{E}$. The network has N_T sources and N_R sinks. The relays can 1) linearly combine incoming signals and 2) amplify-and-forward signals to the next relay or sink. All the links in the network are noisy, we define $\boldsymbol{\eta} \in \mathbb{C}^{N_\eta}$ as the noise vector in the network, with $\boldsymbol{\eta} \sim \mathcal{N}_\mathbb{C}(0, \mathbf{C}_{\boldsymbol{\eta}})$, N_η the number of noisy connections and $\mathbf{C}_{\boldsymbol{\eta}} \in \mathbb{C}^{N_\eta \times N_\eta}$ the covariance matrix of $\boldsymbol{\eta}$.

Each source j must communicate a vector $\tilde{\mathbf{z}}_j \in \mathbb{C}^{n'}$ to the sinks (we assume that all sources send a signal of the same dimension for simplicity). We define a vector $\mathbf{z} = (\tilde{\mathbf{z}}_1, \ldots, \tilde{\mathbf{z}}_{N_T})$ of size $n = N_T n'$ which is the global transmitted vector through the network. Each sink wants to obtain the data from all sources, i.e., to decode $\mathbf{z}$. Each sink i actually decodes a vector $\hat{\mathbf{z}}_i \in \mathbb{C}^n$. A given input signal $\mathbf{z}$ is transmitted successfully if for $i = 1, \ldots, N_R$ we have $\hat{\mathbf{z}}_i = \mathbf{z}$.

Before actually transmitting $\tilde{\mathbf{z}}_j$ each source j has the opportunity to map $\tilde{\mathbf{z}}_j$ to another signal $\tilde{\mathbf{x}}_j$ of different dimension m'_T (it will become clear later why to use such a mapping),i.e., $\tilde{\mathbf{x}}_j = f_j(\tilde{\mathbf{z}}_j)$. Further we define the function f as a mapping from $\mathbb{C}^n$ to $\mathbb{C}^{m_T}$ such that

$$\mathbf{x} = f(\mathbf{z}) = (f_1(\tilde{\mathbf{z}}_1), \ldots, f_{N_T}(\tilde{\mathbf{z}}_{N_T})), \quad (5)$$

where $m_T = N_T m'_T$. Finally the signal $\mathbf{x}$ is sent through the network.

To describe the physical structure of the network we define m_{Ri} as the min-cut between all sources and a sink i. In other words m_{Ri} is the number of disjoint paths between a cluster containing all sources and a sink i.

Definition 1. We define m_R the min-cut of the network as

$$m_R = \inf_{i=1,\ldots,N_R} (m_{Ri}). \quad (6)$$

The min-cut of the network m_R is the smallest number of disjoint paths between all sources and each individual sinks.

We proved in [4] that communication in such a multi-source multi-sink network can be modeled as

$$\hat{\mathbf{x}}_i = \mathbf{B}_i^{\mathrm{H}} \mathbf{MAs}. \quad (7)$$

where the vector $\hat{\mathbf{x}}_i \in \mathbb{C}^{m_R}$ is the received signal at the sink i (we assume that the sinks receive a signal of the maximum size possible, i.e., m_R), the vector $\mathbf{s} \in \mathbb{C}^{m_T+N_\eta}$ has the form $\mathbf{s} = [\mathbf{x}^\mathrm{T}\ \boldsymbol{\eta}^\mathrm{T}]^\mathrm{T}$, $\mathbf{A} \in \mathbb{C}^{e \times (m_T+N_\eta)}$ represents the amplification factors chosen by the sources, $\mathbf{B}_i \in \mathbb{C}^{e \times m_R}$ the filtering factors chosen by the sink i and finally $\mathbf{M} \in \mathbb{C}^{e \times e}$ is defined as

$$\mathbf{M} = \mathbf{I}_e + \mathbf{F} + \mathbf{F}^2 + \cdots + \mathbf{F}^{p-1} = (\mathbf{I}_e - \mathbf{F})^{-1}, \quad (8)$$

where $\mathbf{F} \in \mathbb{C}^{e \times e}$ contains the amplification factors chosen by the relays and $p-1$ is the maximum number of hops from a source to a sink ($\mathbf{F}$ is nilpotent of degree p). Further $\mathbf{M}$ can be represented as a block matrix with the following structure

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{\mathbf{L}1} & \ldots & \mathbf{M}_{\mathbf{L}N_T} & \mathbf{M}_{\mathbf{R}} \\ \mathbf{M}_{\mathbf{x}1,1} & \ldots & \mathbf{M}_{\mathbf{x}1,N_T} & \mathbf{M}_{\boldsymbol{\eta}1} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{M}_{\mathbf{x}N_R,1} & \ldots & \mathbf{M}_{\mathbf{x}N_R,N_T} & \mathbf{M}_{\boldsymbol{\eta}N_R} \end{bmatrix}, \quad (9)$$

with $\mathbf{M}_{\mathbf{x}i,j} \in \mathbb{C}^{m_R \times m'_T}$ is a matrix containing the amplification factors experienced by $\tilde{\mathbf{x}}_j$ on the way to the

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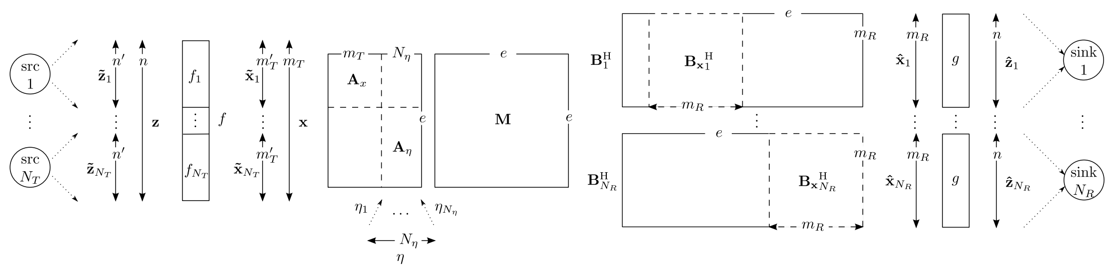

Fig. 1: Overview of the model notations, N_T, N_R and N_η are respectively the number of sources, sinks and noise components, $\tilde{\mathbf{z}}_j$ is the input vector of size n' at source j, $\mathbf{z}$ is the global input vector of size n at all sources, $f = (f_1, \ldots, f_{N_T})$ is a mapping from $\mathbb{C}^n$ to $\mathbb{C}^{m_T}$, $\tilde{\mathbf{x}}_j$ is the transmitted signal of size m'_T from source j, $\mathbf{x}$ is the global transmitted signal of size m_T from all sources, $\hat{\mathbf{x}}_i$ is the received signal of size m_R at sink i, g is a decoder mapping $\mathbb{C}^{m_R}$ to $\mathbb{C}^n$, $\hat{\mathbf{z}}_i$ is the decoded vector of size n at sink i, $\boldsymbol{\eta}$ is the noise vector of size N_η, $\mathbf{A}$ distributes input signals into the network, $\mathbf{M}$ represents the network, $\mathbf{B}_i$ gathers output signals at sink i and e is the number of edges in the network.

sink i, $\mathbf{M}_{\mathbf{L}j} \in \mathbb{C}^{(e-N_R m_R)\times m'_T}$ contains amplification factors of $\tilde{\mathbf{x}}_j$ to the relays, $\mathbf{M}_{\mathbf{R}} \in \mathbb{C}^{(e-N_R m_R)\times N_\eta}$ contains amplification factors of the noise to the relays and finally $\mathbf{M}_{\boldsymbol{\eta}i} \in \mathbb{C}^{m_R \times N_\eta}$ contains amplification factors of the noise to the sink i. As explained in [4], the matrix $\mathbf{A}$ can be represented as a block matrix with the following structure

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{\mathbf{x}} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{\boldsymbol{\eta}} \end{bmatrix}, \tag{10}$$

where $\mathbf{A}_{\mathbf{x}} \in \mathbb{C}^{m_T \times m_T}$ is a block diagonal precoding matrix applied by the sources to the transmitted signal, $\mathbf{A}_{\boldsymbol{\eta}} \in \mathbb{C}^{(e-m_T)\times N_\eta}$ is a matrix composed of zeros and ones which distributes the noise on the edges of the network. By defining $\mathbf{M}_{\mathbf{x}i} = [\mathbf{M}_{\mathbf{x}i,1}, \ldots, \mathbf{M}_{\mathbf{x}i,N_T}]$ we can further develop (7) and achieve the following expression

$$\hat{\mathbf{x}}_i = \mathbf{B}_{\mathbf{x}i}^{\mathrm{H}}(\mathbf{M}_{\mathbf{x}i}\mathbf{A}_{\mathbf{x}}\mathbf{x} + \mathbf{M}_{\boldsymbol{\eta}i}\mathbf{A}_{\boldsymbol{\eta}}\boldsymbol{\eta}), \tag{11}$$

where $\mathbf{B}_{\mathbf{x}i} \in \mathbb{C}^{m_R \times m_R}$ (a submatrix of $\mathbf{B}_i$) is a filter applied by the sink i to the received signal. After receiving $\hat{\mathbf{x}}_i$ a sink i applies a reconstruction algorithm g, a mapping from $\mathbb{C}^{m_R}$ to $\mathbb{C}^n$, to recover $\mathbf{z}$. Formally the recovery is successful if

$$\hat{\mathbf{z}}_i = g(\mathbf{B}_{\mathbf{x}i}^{\mathrm{H}}(\mathbf{M}_{\mathbf{x}i}\mathbf{A}_{\mathbf{x}}f(\mathbf{z}) + \mathbf{M}_{\boldsymbol{\eta}i}\mathbf{A}_{\boldsymbol{\eta}}\boldsymbol{\eta})) = \mathbf{z}, \tag{12}$$

for all sinks i with $i = 1, \ldots, N_R$. Further each individual source is power-limited which is represented by the following constraint

$$\mathbb{E}[\|\mathbf{A}_{\mathbf{x}}[0\ \ldots\ \tilde{\mathbf{x}}_j\ \ldots\ 0]^{\mathrm{H}}\|_{l_2}^2] \leq P_{Tj} = P_T, \tag{13}$$

with P_T the available power at each source. To make the notations clearer we illustrate the system model in Fig. 1.

IV. Compressive Forwarding

The central idea of this paper is as follow: if we don't assume any structure on the input signal $\mathbf{z}$, then a recovery as illustrated in (12), requires to have $n = m_T = m_R$, i.e., the dimension of the transmitted signal must be the same as the min-cut of the network. However if we assume the input signal to have a specific structure we can transmit a signal of dimension larger than the physical size of the network by using the compressed sensing methodology to compress and recover $\mathbf{z}$. In following we will assume the signals $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{N_T}$ to be jointly sparse.

Definition 2. We call several signals $\tilde{\mathbf{z}}_1, \tilde{\mathbf{z}}_2, \ldots, \tilde{\mathbf{z}}_{N_T}$ jointly k-sparse if the vector $\mathbf{z} = (\tilde{\mathbf{z}}_1, \tilde{\mathbf{z}}_2, \ldots, \tilde{\mathbf{z}}_{N_T})$ is k-sparse (it has no more than k nonzero components).

Note that this joint sparsity model is a simple subset of the JSM-1 model of [9], which represents the fact that the sources communicate correlated information, modeled as sparse, to the sinks. There exist much more joint sparsity models (e.g., JSM-2), and the concepts of the present work apply to them.

The main methodology of this paper is the following: we need to optimize $\mathbf{B}_{\mathbf{x}i}, \mathbf{M}_{\mathbf{x}i}, \mathbf{A}_{\mathbf{x}}, \mathbf{M}_{\boldsymbol{\eta}i}, \mathbf{A}_{\boldsymbol{\eta}}, f$ and g to enable $\hat{\mathbf{z}}_i = \mathbf{z}$ with $n \gg m_R$ and $\mathbf{z}$ being k-sparse. We first neglect the noise in the optimization problem (which corresponds to the high SNR regime) and then evaluate the performance of our solution in presence of noise. In the high SNR regime, Equation (12) modifies to

$$\hat{\mathbf{z}}_i = g(\mathbf{B}_{\mathbf{x}i}^{\mathrm{H}}\mathbf{M}_{\mathbf{x}i}\mathbf{A}_{\mathbf{x}}f(\mathbf{z})) = g(\mathbf{N}_{\mathbf{x}i}f(\mathbf{z})) = \mathbf{z}, \tag{14}$$

with the simplifying notation $\mathbf{N}_{\mathbf{x}i} = \mathbf{B}_{\mathbf{x}i}^{\mathrm{H}}\mathbf{M}_{\mathbf{x}i}\mathbf{A}_{\mathbf{x}}$. Note that since $\mathbf{M}$ is a sum of powers of $\mathbf{F}$, it is difficult to optimize. In the rest of this paper we will draw the coefficients of $\mathbf{F}$ at random, thus fixing $\mathbf{M}$, and optimize $\mathbf{B}_{\mathbf{x}i}, \mathbf{A}_{\mathbf{x}}, f$ and g for several types of network to perform compressive forwarding.

V. Single-Source Multi-Sink Network

In this network, there is a unique source transmitting $\mathbf{z}$ to N_R sinks (each sink needs to recover $\mathbf{z}$). We have $m_T = m'_T$ since there is only one signal to transmit. Further the single source can use for itself the whole dimension offered by the network, i.e., $m_T = m'_T = m_R$ and therefore $\mathbf{M}_{\mathbf{x}i}$ is a square $m_R \times m_R$ matrix.

Theorem 1. *If $\mathbf{M}_{\mathbf{x}i}$ is invertible for $i = 1, \ldots, N_R$, then it is possible to choose $\mathbf{A}_{\mathbf{x}}$ and $\mathbf{B}_{\mathbf{x}i}$, enabling perfect recovery of $\mathbf{z}$ from $\hat{\mathbf{x}}_{\mathbf{i}}$ with $m_R \geq c_1 k \log(n/k)$ where c_1 is a constant.*

Proof: Take $f(\mathbf{z}) = \mathbf{G}\mathbf{z}$, where $\mathbf{G}$ is a $m_R \times n$ matrix with coefficient chosen i.i.d. at random as

$$g_{ij} \sim \mathcal{N}(0, 1/m_R). \tag{15}$$

Each sink receives $\hat{\mathbf{x}}_i = \mathbf{N}_{\mathbf{x}i}\mathbf{G}\mathbf{z}$. We normalize the power of the signal $\mathbf{z}$ as $\|\mathbf{z}\|_{l_2}^2 = 1$ and it can be shown that the power constraint (13): $\|\mathbf{A}_{\mathbf{x}}\mathbf{x}\|_{l_2}^2 \leq P_T$, is equivalent to

$$\mathrm{Tr}(\mathbf{A}_{\mathbf{x}}\mathbf{A}_{\mathbf{x}}^{\mathrm{H}}) \leq m_R P_T. \tag{16}$$

Take $\mathbf{A_x} = \sqrt{P_T}\ \mathbf{I}_{m_R}$, then the power constraint is fulfilled. Then take $\mathbf{B}_{\mathbf{x}i} = \mathbf{I}_{m_R}$ and $\mathbf{N}_{\mathbf{x}i}$ is invertible since per assumption $\mathbf{M}_{\mathbf{x}i}$ is invertible. Now because all $\mathbf{N}_{\mathbf{x}i}$'s are invertible, each sink can compute $\hat{\mathbf{z}}_i$ with the following decoding algorithm g

$$\begin{array}{ll} \text{minimize} & \|\hat{\mathbf{z}}_i\|_{l_1} \\ \text{subject to} & \mathbf{G}\hat{\mathbf{z}}_i = \mathbf{N}_{\mathbf{x}i}^{-1}\hat{\mathbf{x}}_i \end{array} \tag{17}$$

If $m_R \geq c_1 k \log(n/k)$, with c_1 a constant, then it is shown in [6] that $\mathbf{G}$ satisfies the RIP of order k with probability greater than $1 - e^{c_2 m_R}$ (c_2 a constant). Therefore the optimization problem (17) recovers $\mathbf{z}$ exactly for all sinks, which concludes the proof. ■

Note that simply choosing the coefficients of $\mathbf{F}$ in (8) at random would lead to a have $\mathbf{M}_{\mathbf{x}i}$ almost surely invertible.

Obviously the multiplication of $\mathbf{z}$ with $\mathbf{G}$ is equivalent to first compress $\mathbf{z}$ and then send it through the network. This simple network structure (single source) does not demonstrate distributed compression but we presented it because it still gives understanding about the overall problem. A more challenging case is when the source signal $\mathbf{z}$ is distributed among multiples sources and therefore no global compression can be performed at the sources.

VI. Multi-Source Networks

In this section we consider networks with N_T sources. It is not possible to multiply $\mathbf{z}$ with a complete $m_R \times n$ Gaussian matrix since the sources are independent and have only access to their own signal. We first define a source transmission strategy that we will use later on for our results.

Definition 3. We call *single-source min-cut rate* the maximum dimension of a source signal when each source transmits as if it was the only source in the network. If m_R is the min-cut of the network each source sends a signal of dimension m_R.

In other word the single-source min-cut rate is achieved by superposing the signal of all sources on the m_R disjoint paths to the sinks. Note that when the sources transmit at the single-source min-cut rate, the matrices $\mathbf{N}_{\mathbf{x}i}$ are of size $m_R \times N_T m_R$ and the matrices $\mathbf{M}_{\mathbf{x}i,j}$ are square of size $m_R \times m_R$. In the following we first consider the simpler case with only one sink in the network and secondly the more involved case of N_R sinks.

A. Single-Sink Networks

In this problem all sources can optimize their transmission to help a single sink.

Theorem 2. *If all sources transmit at the single-source min-cut rate and all matrices $\mathbf{M}_{\mathbf{x}1,j}$ corresponding to the amplification factors experienced by $\mathbf{z}_j$ on the way to the single sink (indexed 1) are invertible for $j = 1, \ldots, N_T$, then it is possible to choose $\mathbf{A_x}$ and $\mathbf{B_x}$, enabling perfect recovery of $\mathbf{z}$ from $\hat{\mathbf{x}}$ (we delete the index as there is a single sink) with $m_R \geq c_1 k \log(n/k)$ where c_1 is a constant.*

Proof: We let all sources send at the single-source min-cut rate. The matrix $\mathbf{A_x}$ has the form

$$\mathbf{A_x} = \begin{bmatrix} \mathbf{A}_{\mathbf{x}1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{A}_{\mathbf{x}N_T} \end{bmatrix}, \tag{18}$$

with $\mathbf{A}_{\mathbf{x}j}$ a $m_R \times m_R$ matrix. We choose $f(\mathbf{z}) = \mathbf{Gz}$ with $\mathbf{G}$ a block diagonal matrix with the following structure

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{G}_{N_T} \end{bmatrix}, \tag{19}$$

with $\mathbf{G}_j$ a $m_R \times n'$ matrix with coefficients chosen i.i.d. at random as $[G_j]_{a,b} \sim \mathcal{N}(0, 1/m_R)$. In other word each source multiplies its part of the vector $\mathbf{z}$ by a Gaussian matrix, i.e., performing local compression. We will use the network to perform global compression. Equation (14) becomes $\hat{\mathbf{x}} = \mathbf{N_x G z}$ (note that we dropped again the index of $\hat{\mathbf{x}}$ and $\mathbf{N_x}$ since there is a singles sink). Further this expression can be detailed as

$$\hat{\mathbf{x}} = \left[\mathbf{B_x}^{\mathrm{H}}\mathbf{M}_{\mathbf{x}1,1}\mathbf{A}_{\mathbf{x}1}\mathbf{G}_1 \ldots \mathbf{B_x}^{\mathrm{H}}\mathbf{M}_{\mathbf{x}1,N_T}\mathbf{A}_{\mathbf{x}T}\mathbf{G}_{N_T}\right]\mathbf{z}. \tag{20}$$

The most important thing to see here is that $\mathbf{N_x G}$ is composed of N_T sub-matrices which are distorted Gaussian matrices, in the sense that they are not anymore i.i.d. with mean zero and variance $1/m_R$. If we could eliminate these distortions, $\mathbf{N_x G}$ would be a $m_R \times n$ Gaussian matrix. To reach this goal, we can optimize $\mathbf{B_x}$, $\mathbf{M}_{\mathbf{x}j}$ and $\mathbf{A}_{\mathbf{x}j}$. As already mentioned it is not possible to directly access the coefficients of $\mathbf{M}_{\mathbf{x}j}$ (we can choose the coefficients of $\mathbf{F}$ which then enter in $\mathbf{M}_{\mathbf{x}j}$ as in (8)). We therefore choose all its coefficients at random and try to eliminate the distortions by choosing only $\mathbf{B_x}$ and $\mathbf{A}_{\mathbf{x}j}$.

The power constraint (13) in that case can be expressed as $\mathbb{E}(\|\mathbf{A}_{\mathbf{x}j}\tilde{\mathbf{x}}_j\|_{l_2}^2) \leq P_T$, which is equivalent to

$$\mathrm{Tr}(\mathbf{A}_{\mathbf{x}j}\mathbf{A}_{\mathbf{x}j}^{\mathrm{H}}) \leq m_R P_T. \tag{21}$$

Take

$$\mathbf{A}_{\mathbf{x}j} = \sqrt{m_R P_T}\,\frac{\mathbf{M}_{\mathbf{x}1,j}^{-1}}{\max\limits_{j=1,\ldots,N_T} \|\mathbf{M}_{\mathbf{x}1,j}^{-1}\|_F}, \tag{22}$$

where $\|.\|_F$ denotes the Frobenius norm and

$$\mathbf{B_x} = \frac{\max\limits_{j=1,\ldots,N_T} \|\mathbf{M}_{\mathbf{x}1,j}^{-1}\|_F}{\sqrt{m_R P_T}}\mathbf{I}_{m_R} \tag{23}$$

then the power constraint is fulfilled for $j = 1, \ldots, N_T$, $\mathbf{N_x} = [\mathbf{I}_{m_R} \ \ldots \ \mathbf{I}_{m_R}]$ and $\mathbf{N_x G}$ is now a Gaussian matrix of size $m_R \times n$ which verifies the RIP of order k. It only remains to solve the following optimization problem g

$$\begin{array}{ll} \text{minimize} & \|\hat{\mathbf{z}}\|_{l_1} \\ \text{subject to} & \mathbf{N_x G}\hat{\mathbf{z}} = \hat{\mathbf{x}} \end{array} \tag{24}$$

to recover $\mathbf{z}$ from $\hat{\mathbf{x}}$. ■

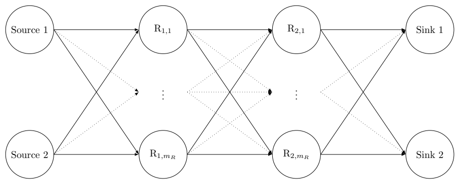

Fig. 2: Example of relay network

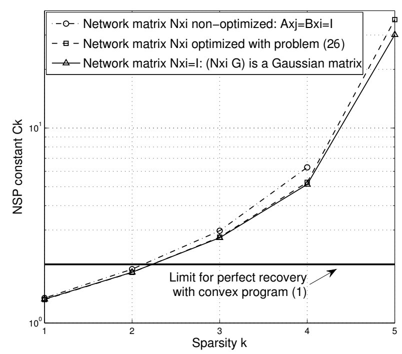

Fig. 3: Evolution of the NSP w.r.t. k

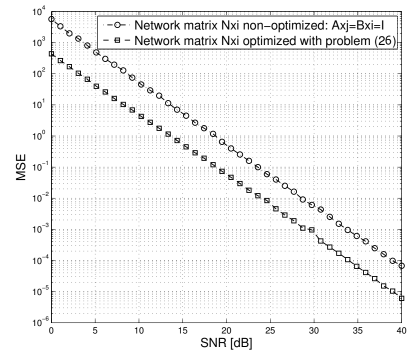

Fig. 4: Average MSE at the sinks for $k = 3$

B. Multi-Sink Networks

In a multiple-sink network, $\hat{\mathbf{x}}_i$ can be written as

$$\hat{\mathbf{x}}_i = \left[\mathbf{B}_{\mathbf{x}_i}^{\mathrm{H}}\mathbf{M}_{\mathbf{x}i,1}\mathbf{A}_{\mathbf{x}1}\mathbf{G}_1 \ldots \mathbf{B}_{\mathbf{x}_i}^{\mathrm{H}}\mathbf{M}_{\mathbf{x}i,N_T}\mathbf{A}_{\mathbf{x}T}\mathbf{G}_T\right]\mathbf{z}. \quad (25)$$

In this type of network we need to eliminate distortions at all sinks, the sources cannot optimize their transmission for only one sink but must consider several network paths to different sinks. If a matrix is i.i.d Gaussian, then it is unitarily invariant [12]. Formally $\mathbf{U}_j\mathbf{G}_j \sim \mathbf{G}_j$ if $\mathbf{U}_j \in \mathbb{R}^{m_R \times m_R}$ is unitary.

So that $\mathbf{N}_{\mathbf{x}i}\mathbf{G}$ is a Gaussian i.i.d. matrix with zero-mean and variance $1/m$ for $i = 1, \ldots, N_R$, we need to choose $\mathbf{B}_{\mathbf{x}i}$ and $\mathbf{A}_{\mathbf{x}j}$ for $i = 1, \ldots, N_R$ and $j = 1, \ldots, N_T$ such that $\mathbf{N}_{\mathbf{x}i}$ is unitary for $i = 1, \ldots, N_R$. Note that as in the previous section, we draw all coefficients of $\mathbf{M}_{\mathbf{x}i,j}$ at random. Formally this is equivalent to have $\mathbf{B}_{\mathbf{x}_i}^{\mathrm{H}}\mathbf{M}_{\mathbf{x}i,j}\mathbf{A}_{\mathbf{x}j}\mathbf{A}_{\mathbf{x}_j}^{\mathrm{H}}\mathbf{M}_{\mathbf{x}_{i,j}}^{\mathrm{H}}\mathbf{B}_{\mathbf{x}i} = \mathbf{I}_{m_R}$ for $i = 1, \ldots, N_R$ and $j = 1, \ldots, N_T$. This goal can be formulated as the following optimization problem

$$\begin{array}{ll} \text{minimize} & \gamma^2 \\ \text{subject to} & \mathbf{Q}_i \preccurlyeq \mathbf{M}_{\mathbf{x}i,j}\mathbf{P}_j\mathbf{M}_{\mathbf{x}_{i,j}}^{\mathrm{H}} \preccurlyeq \gamma^2\mathbf{Q}_i \\ & \mathbf{Q}_i \succ 0,\ \mathbf{P}_j \succ 0,\ \mathrm{Tr}(\mathbf{P}_j) \leq m_R P_T \\ & 1 \leq i \leq N_R,\ 1 \leq j \leq N_T \end{array} \quad (26)$$

with γ a scalar variable, $\mathbf{A}_{\mathbf{x}j} = \mathbf{P}_j^{1/2}$ and $\mathbf{B}_{\mathbf{x}i} = \mathbf{Q}_i^{-1/2}$. Problem (26) is a generalized eigenvalue problem (GEVP) [13] which is quasiconvex and can be easily solved using the bisection method. Similarly to the single-sink case, by solving

$$\begin{array}{ll} \text{minimize} & \|\hat{\mathbf{z}}_{\mathbf{i}}\|_{l_1} \\ \text{subject to} & \mathbf{N}_{\mathbf{x}i}\mathbf{G}\hat{\mathbf{z}}_{\mathbf{i}} = \hat{\mathbf{x}}_{\mathbf{i}}, \end{array} \quad (27)$$

each sink i can recover $\mathbf{z}$.

VII. Simulation Results

In this section we numerically evaluate the concept presented above on a network (illustrated in Fig. 2) with two sources, two sinks and two layers of m_R relays each. We choose $m_T = m_R = 20$ and $n = 30$, i.e., we transmit a vector $\mathbf{z}$ with a dimension 50% larger than the physical size of the network. The relay amplification factors are drawn i.i.d. at random $\sim \mathcal{N}(0, 1)$. We solve the optimization problem (26) and (27) using CVX [14]. In Fig. 3, we plot the average NSP of $\mathbf{N}_{\mathbf{x}i}\mathbf{G}$ with respect to the sparsity of $\mathbf{z}$. We see that whereas the network matrix optimized with problem (26) stay closed to optimal NSP of a Gaussian matrix, the non-optimized matrix rapidly diverges (for $k = 5$, there exists no algorithm that enables perfect recovery). In Fig. 4 we plot the average mean square error (MSE) between $\mathbf{z}$ and $\hat{\mathbf{z}}_{\mathbf{i}}$. The optimization problem (26) enables a gain of one order of magnitude compared to the non-optimized case.

VIII. Conclusion

We have presented a framework to perform forwarding in relay networks using compressed sensing as distributed compression algorithm. Further we developed and evaluated methods to optimize the network matrices to improve the network throughput when the transmitted signals are jointly sparse.

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