Cyclic Interference Alignment by Propagation Delay

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***Abstract*—In the present paper, Interference Alignment by propagation delay is applied to a delay-based X-channel and a delay-based K-user interference channel. The key ingredient in our approach is the cyclic permutation property of the delay-based channel model that uses cyclic right-shifts in polynomials.**

Based on this cyclic channel model, we derive necessary conditions on the propagation delay matrix between users and provide *Cyclic Interference Alignment* schemes achieving the upper bounds on X-networks as given by Cadambe et al.

By further assuming that the propagation delays are proportional to the Euclidean distances, a delay matrix with integer-valued propagation delays can be derived. It enables us to investigate the placement of user-nodes in Euclidean space such that Cyclic Interference Alignment is achieved in two dimensions.

I. Introduction

Considering the challenging problem to mitigate mutual interference in wireless multi-user communication systems, *Interference Alignment* (IA) has emerged in [1], [2] as an innovative concept to efficiently exploit the *degrees of freedom* (DoF) of the signal space in order to achieve high data rates.

The main idea of IA is to overlap interference of multiple users as if each receiver would only perceive a single virtual interferer. Hence, the data rates are not limited by the number of interfering users. An extensive overview on the diversity of different schemes for IA is given in the survey [3].

IA by propagation delay has originally been introduced in the seminal work [1]. However, it is only presented as a simple toy example to illustrate the elementary ideas behind the novel technique. A comparable toy example for an X-channel is also provided in [4]. Apart from those basic examples, a closer look at IA by propagation delay is taken in [5] and to some extend in [6]–[8].

In [5], [9]–[11], the authors further assume that propagation delay is proportional to the Euclidean distance between transmitters and receivers and derive *node placement* schemes, i. e., to answer the question of how to position transmitters and receivers in spatial dimensions to achieve IA by propagation delay.

An evaluation of the performance of IA by propagation delay for users placed completely at random is conducted in [12]. An opportunity for practical applications in satellite and submarine networks is mentioned in [12]–[14].

Furthermore, the authors of [11] consider IA by propagation delay and node placement for cognitive radio systems.

Contributions. In this work, we introduce a system model for IA by propagation delay that operates with polynomials that describe cyclic permutations. The mathematical framework is inspired by *cyclic codes*.

Our main goal is to formally generalize IA by propagation delay on delay-based interference channels as treated in [1], [4], [5], [9], [10]. We intend to keep an intuitive representation of the considered model as in the original examples.

To ensure decodability in the presence of interference, we derive *separability conditions* and a sufficient *Cyclic IA* scheme for the special cases of delay-based X-*channels* and K-*user interference channels*. We derive an upper bound on the DoF for delay-based interference channels with an arbitrary number of messages and users in analogy to [4, Thm. 1].

Furthermore, we present a solution for node placement of four users in 2-dimensional Euclidean space to realize the discrete delay-based channel matrices of an X-channel.

Organization. In Section II, we introduce the cyclic representation for a delay-based model of the X-channel. The Cyclic IA scheme for the X-channel is investigated in Section III. A resulting node placement in two dimensions is elaborated in Section IV. Cyclic IA on the K-user interference channel is investigated in Section V. An upper bound on the DoF is provided in Section VI. We discuss further insights on Cyclic IA in Section VII and give a conclusion in Section VIII.

Notation. Matrices are denoted by boldface capital letters and vectors by boldface lower-cases. Determinants are denoted by $\det(\cdot)$ and transposed matrices by $(\cdot)^{\mathsf{T}}$. $\boldsymbol{I}_n$ is the $n \times n$ identity matrix, $\mathbf{1}_{m\times n}$ the $m \times n$-matrix of ones, and $\mathbf{0}_n$ the n-dimensional zero vector. A univariate polynomial of degree n in the indeterminate x is denoted by $p(x) = \sum_{i=0}^{n-1} p^{[i]} x^i$ with coefficients $p^{[i]}$.

II. System Model

The instructive toy example given in [4] considers an X-channel with a specific set of discrete propagation delays. However, the given example does not elaborate how to adjust the scheme to a channel with arbitrary discrete delays. We intend to formally specify the delay-based channel model and the corresponding IA scheme in the following.

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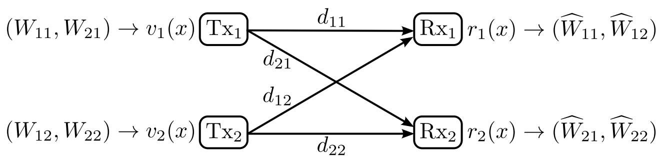

Fig. 1. The delay-based X-channel with messages W_{11}, W_{12}, W_{21} and W_{22} between Tx_1, Tx_2, Rx_1 and Rx_2 and delays d_{11}, d_{12}, d_{21} and d_{22} of delay matrix $\boldsymbol{D}$.

An X-*channel* describes a wireless channel with $K = 4$ users, i. e., 2 transmitters and 2 receivers, as depicted in Fig. 1. We denote a message between a transmitter Tx_i and a receiver Rx_j, $i, j \in \mathcal{K} := \{1, 2\}$, by W_{ji}. An X-channel is physically equivalent to an interference channel albeit with a number of $M = 4$ independent messages W_{11}, W_{21}, W_{12} and W_{22} instead of only $M = 2$ independent messages W_{11} and W_{22}.

The channel access at each Tx_i and Rx_j is partitioned into $n \in \mathbb{N}$ equally sized time-slots, each normalized to length one. A transmitter Tx_i can allocate one message W_{ji} per time-slot.

Propagation delays between any Tx_i and Rx_j are assumed to be static and non-negative integer multiples of one time-slot. These discrete delays could be established through a proper node placement as we will discuss in Section IV.

Like in conventional orthogonal multiple-access schemes, the channel access repeats itself after n time-slots and new messages are transmitted each period. Firstly, there is a transient settling time determined by the longest propagation delay in the channel. Then, the channel access is stationary over a period of n consecutive time-slots. Within the stationary period, delayed messages are cyclically right-shifted.

We model such a communication system with cyclic right-shifts by polynomials in x modulo $x^n - 1$. Single time-slots in the period of n time-slots are addressed by *offsets* $x^0, x^1, \ldots, x^{n-1}$, from 0 (no offset) to $n-1$ (maximal offset). The propagation delay between a pair (Rx_j,Tx_i) is denoted by $d_{ji} \in \mathcal{D} = \{x^k | k \in \mathbb{N}\}$. The matrix of propagation delays is assumed to be fully known to all users and defined by:

$$\boldsymbol{D} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}. \tag{1}$$

Let a message (coefficient) W be transmitted at offset x^u and delayed by v time-slots. The resulting delayed message is computed by $W x^{u+v} \bmod (x^n - 1)$ for a period of n time-slots.

Encoding Scheme: The codeword sent from Tx_1 is encoded into the polynomial $v_1(x)$ by the encoding function e_1 carrying the two messages W_{11}, W_{21}. Accordingly, codeword $v_2(x)$ at Tx_2 is encoded by e_2 carrying W_{12}, W_{22}:

$$e_1 : (W_{11}, W_{21}) \to v_1(x),$$
$$e_2 : (W_{12}, W_{22}) \to v_2(x).$$

The (cyclic) transfer function is described by the superposition of input polynomials in vector $\boldsymbol{v} = (v_1(x), v_2(x))$ s. t. the vector of received polynomials $\boldsymbol{r} = (r_1(x), r_2(x))$ yields:

$$\boldsymbol{r}^\top \equiv \boldsymbol{D} \boldsymbol{v}^\top \bmod (x^n - 1). \tag{2}$$

It is easily observed that undesired messages cause (cyclic) interference at both receivers.

	x^0	x^1	x^2
$\mathrm{Tx}_1 : v_1(x)$	W_{11}	0	W_{21}
$\mathrm{Tx}_2 : v_2(x)$	0	W_{12}	W_{22}
$\mathrm{Rx}_1 : r_1(x)$	$W_{21}+W_{22}$	W_{11}	W_{12}
$\mathrm{Rx}_2 : r_2(x)$	$W_{11}+W_{12}$	W_{22}	W_{21}

Fig. 2. Cyclic IA is performed on the delay-based X-channel in a period of $n = 3$ time-slots. The following delays $d_{11} = x^1$, $d_{12} = x^1$, $d_{21} = x^3$ and $d_{22} = x^2$ are assumed. The transmitted polynomials are chosen as $v_1(x) = W_{11} + x^2 W_{21}$ and $v_2(x) = x W_{12} + x^2 W_{22}$. The received polynomials yield $r_1(x) = (W_{21}+W_{22})+xW_{11}+x^2 W_{12}$ and $r_2(x) = (W_{11}+W_{12})+xW_{22}+x^2 W_{21}$. The $M = 4$ desired messages are decoded by taking $r_1^{[1]} = \widehat{W}_{11}, r_1^{[2]} = \widehat{W}_{12}, r_2^{[2]} = \widehat{W}_{21}$ and $r_2^{[1]} = \widehat{W}_{22}$.

Decoding Scheme: The received polynomials are decoded to obtain an estimate of the dedicated messages $\widehat{W}_{ji}$. The received polynomial $r_1(x)$ is decoded by f_1 and $r_2(x)$ by f_2:

$$f_1 : r_1(x) \to (\widehat{W}_{11}, \widehat{W}_{12}),$$
$$f_2 : r_2(x) \to (\widehat{W}_{21}, \widehat{W}_{22}).$$

An example of a transmission scheme within one (stationary) period of $n = 3$ time-slots is shown in Fig. 2.

Note that in the linear decoding process the cyclic delays are unrolled over time. The decoder must also take the transient settling time at the beginning into account.

Assuming i.i.d. zero mean unit variance Gaussian noise at the receivers and an average power constraint P per message within each time-slot, a single interference-free link between Tx_i and Rx_j has a capacity of $\log(P)$ bits per time-slot at high SNR. The *degrees of freedom* (DoF) metric is defined in [1], [4] as the pre-log of the achieved sum-capacity $C_\Sigma(P)$:

$$\mathrm{DoF} = \lim_{P \to \infty} \frac{C_\Sigma(P)}{\log(P)}.$$

In the delay-based model, the achieved DoF are the number of messages M received interference-free per n time-slots:

$$\mathrm{DoF} = \lim_{P \to \infty} \frac{\frac{M}{n} \log(P)}{\log(P)} = \frac{M}{n}. \tag{3}$$

III. Cyclic Interference Alignment

For the delay-based X-channel, the task of Cyclic IA is to convey and decode $M = 4$ dedicated messages W_{11}, W_{12}, W_{21} and W_{22} interference-free in a period of n time-slots. The scheme is optimal in the sense of Cyclic IA if n is minimal and still feasible.

We call interfering messages to be *aligned* if at least two of these are received in the same time-slot at an undesired receiver. In order to align the messages W_{22} and W_{21} in $v_1(x)$ and $v_2(x)$, they must overlap in the same time-slot at Rx_1 but remain distinct in different time-slots at Rx_2. To align the messages W_{11} and W_{12} in $v_1(x)$ and $v_2(x)$, they must overlap at Rx_2 but remain distinct at Rx_1, accordingly. An example of such an alignment is shown in Fig. 2. We choose the following polynomials for transmission using the parameters $p_{11}, p_{12}, p_{21}, p_{22} \in \mathbb{N}_0$:

$$v_1(x) = W_{11} x^{p_{11}} + W_{21} x^{p_{21}}, \tag{4}$$
$$v_2(x) = W_{12} x^{p_{12}} + W_{22} x^{p_{22}}. \tag{5}$$

Then by (2), the received polynomials at Rx_1 and Rx_2 yield:

$$\begin{aligned} r_1(x) \equiv\ & d_{11}W_{11}x^{p_{11}} + d_{12}W_{12}x^{p_{12}} + \\ & d_{11}W_{21}x^{p_{21}} + d_{12}W_{22}x^{p_{22}} \mod(x^n-1), \\ r_2(x) \equiv\ & d_{21}W_{21}x^{p_{21}} + d_{22}W_{22}x^{p_{22}} + \\ & d_{21}W_{11}x^{p_{11}} + d_{22}W_{12}x^{p_{12}} \mod(x^n-1). \end{aligned}$$

Both messages dedicated for Rx_j must be received in separate time-slots. We term this as the *multiple-access interference condition*. With the indices $j \in \mathcal{K}$, $i \neq l \in \mathcal{K}$, it holds:

$$d_{ji}x^{p_{ji}} \not\equiv d_{jl}x^{p_{jl}} \mod(x^n-1). \tag{6}$$

The two messages from Tx_i dedicated for different receivers $\mathrm{Rx}_j, \mathrm{Rx}_k$ must also be separate. We term this as the *intra-user interference condition* with $j \neq k \in \mathcal{K}$, $i \in \mathcal{K}$:

$$x^{p_{ji}} \not\equiv x^{p_{ki}} \mod(x^n-1). \tag{7}$$

The messages desired at the respective receiver must be separate from both interfering messages. Hence, the *inter-user interference condition* holds with $j \neq k \in \mathcal{K}$, $i \neq l \in \mathcal{K}$:

$$d_{ji}x^{p_{ji}} \not\equiv d_{jl}x^{p_{kl}} \mod(x^n-1). \tag{8}$$

We count ten of such *separability conditions* in total for all valid combinations of indices in $\mathcal{K}$.

For *perfect* Cyclic IA, all interfering messages must overlap in one time-slot. In the case of the X-channel, intra- and inter-user interference is aligned to a single time-slot at each Rx_j:

$$d_{ji}x^{p_{ki}} \equiv d_{jl}x^{p_{kl}} \mod(x^n-1), \tag{9}$$

with $j \neq k \in \mathcal{K}$ and $i \neq l \in \mathcal{K}$.

We remark that (9) substituted into (8) yields (7):

$$\begin{aligned} & d_{ji}x^{p_{ji}} \not\equiv d_{jl}x^{p_{kl}} \mod(x^n-1) \\ \Rightarrow\ & d_{ji}x^{p_{ji}} \not\equiv d_{ji}x^{p_{ki}} \mod(x^n-1) \\ \Rightarrow\ & x^{p_{ji}} \not\equiv x^{p_{ki}} \mod(x^n-1), \end{aligned}$$

with the indices $j \neq k \in \mathcal{K}$ and $i \neq l \in \mathcal{K}$. Hence, we can neglect the condition (8) if both (7) and (9) hold.

Theorem 1. *A perfect Cyclic IA scheme for the delay-based X-channel satisfying the separability conditions exists, if and only if both $\det(\boldsymbol{D}) \not\equiv 0 \mod(x^n-1)$ and $n > 2$ hold. Then, Cyclic IA achieves $\frac{4}{3}$ DoF on the delay-based X-channel.*

Proof:

(a) *Necessity of* $\det(\boldsymbol{D}) \not\equiv 0 \mod(x^n-1)$, $n \in \mathbb{N}$:
Assuming $\det(\boldsymbol{D}) \equiv 0 \mod(x^n-1)$ yields:

$$\begin{aligned} & \det(\boldsymbol{D}) \equiv 0 \mod(x^n-1) \\ \Rightarrow\ & d_{11}d_{22} - d_{21}d_{12} \equiv 0 \mod(x^n-1) \\ \Rightarrow\ & d_{11}d_{22} \equiv d_{21}d_{12} \mod(x^n-1) \\ \Rightarrow\ & d_{ji}d_{kl} \equiv d_{ki}d_{jl} \mod(x^n-1), \end{aligned} \tag{10}$$

with the indices $j \neq k \in \mathcal{K}$ and $i \neq l \in \mathcal{K}$. Including (10) into condition (9) yields:

$$\begin{aligned} & d_{ji}x^{p_{ki}} \equiv d_{jl}x^{p_{kl}} \mod(x^n-1) \\ \Rightarrow\ & d_{ki}x^{p_{ki}} \equiv d_{kl}x^{p_{kl}} \mod(x^n-1). \end{aligned}$$

Relabeling the indices $j \leftrightarrow k$ provides:

$$\Rightarrow d_{ji}x^{p_{ji}} \equiv d_{jl}x^{p_{jl}} \mod(x^n-1),$$

and contradicts (6) for any $n \in \mathbb{N}$.

(b) *Necessity of $n > 2$ time-slots*:
Assume $\det(\boldsymbol{D}) \not\equiv 0 \mod(x^n-1)$ holds. We consider the right-hand sides of (6) and (7).

$$\begin{aligned} (6):\ & d_{ji}x^{p_{ji}} \not\equiv d_{jl}x^{p_{jl}} \mod(x^n-1), \\ (7):\ & d_{ji}x^{p_{ji}} \not\equiv d_{ji}x^{p_{ki}} \mod(x^n-1). \end{aligned}$$

These must also be pair-wise distinct, since we can relabel the indices $i \leftrightarrow l$ in (11) to obtain (8):

$$\begin{aligned} & d_{jl}x^{p_{jl}} \not\equiv d_{ji}x^{p_{ki}} \mod(x^n-1) \\ \Leftrightarrow\ & d_{ji}x^{p_{ji}} \not\equiv d_{jl}x^{p_{kl}} \mod(x^n-1). \end{aligned} \tag{11}$$

Thus, there is no solution to satisfy all three conditions on $d_{ji}x^{p_{ji}}$ with only $n = 1$ or $n = 2$ time-slots.

(c) *Sufficiency of $n = 3$ and $\det(\boldsymbol{D}) \not\equiv 0 \mod(x^n-1)$ to achieve $\frac{4}{3}$ DoF:*
From the perfect IA condition (9), the following holds:

$$x^{p_{12}} \equiv d_{22}^{-1}d_{21}x^{p_{11}} \mod(x^3-1), \tag{12}$$
$$x^{p_{21}} \equiv d_{11}^{-1}d_{12}x^{p_{22}} \mod(x^3-1). \tag{13}$$

Furthermore, the condition (7) must hold:

$$x^{p_{11}} \not\equiv x^{p_{21}} \mod(x^3-1), \tag{14}$$
$$x^{p_{12}} \not\equiv x^{p_{22}} \mod(x^3-1). \tag{15}$$

The insertion of (12) and (13) into condition (14) yields:

$$x^{p_{22}} \not\equiv d_{22}d_{11}d_{21}^{-1}d_{12}^{-1}x^{p_{12}} \mod(x^3-1), \tag{16}$$

Due to (10), the following holds:

$$d_{12}^{-1}d_{21}^{-1}d_{22}d_{11} \not\equiv 1 \mod(x^3-1).$$

W.l.o.g., we can fix p_{11} and compute p_{12} using (12). We can determine a solution for p_{22} from (15) and (16) only if $n > 2$. For $n = 3$ the solution of p_{22} is unique. The remaining parameter p_{21} is derived using (15).
The validity of condition (6) is yet to check. Inserting (12) and (13) into (6) for all cases provides:

$$\begin{aligned} x^{p_{11}} & \not\equiv d_{12}d_{21}d_{22}^{-1}d_{11}^{-1}x^{p_{11}} \mod(x^3-1), \\ x^{p_{21}} & \not\equiv d_{22}d_{11}d_{12}^{-1}d_{21}^{-1}x^{p_{21}} \mod(x^3-1). \end{aligned}$$

Both conditions are satisfied by prerequisite, since $\det(\boldsymbol{D}) \neq 0 \mod(x^3-1)$ holds. Altogether, there is a solution for Cyclic IA on the X-channel with $n = 3$ time-slots and $M = 4$ messages that satisfies the separability conditions and achieves $\frac{4}{3}$ DoF[1]. The result also achieves the upper bound as we will show in Section VI, Lemma 4.

■

Our result also includes the example of the delay-based X-channel considered in [4].

[1]The example solution for Cyclic IA provided in the table of Fig. 2 uses $p_{11} = 0$, $p_{12} = 1$, $p_{21} = 2$, $p_{22} = 2$ and $n = 3$ and achieves $\frac{4}{3}$ DoF.

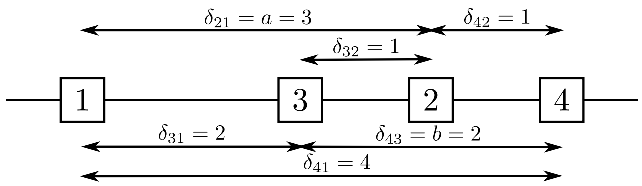

Fig. 3. A one-dimensional solution for the node placement of a given matrix $\boldsymbol{C}$ with $\delta_{31} = 2$, $\delta_{32} = 1$, $\delta_{41} = 4$ and $\delta_{42} = 1$ is shown. The numbered boxes indicate the users. The parameters of the dissimilarity matrix $\boldsymbol{\Delta}$ are $b = 2$ and $a = 3$. Cyclic IA is possible since condition (19) is fulfilled.

Lemma 2. *If* $\det(\boldsymbol{D}) \equiv 0 \bmod(x^n - 1)$ *holds, Cyclic IA achieves only* 1 *DoF on the delay-based* X*-channel.*

Proof: Theorem 1 (a) yields that no two interference signals can be aligned without violating the condition (6), if $\det(\boldsymbol{D}) \equiv 0 \bmod(x^n - 1)$ holds. Thus, each message must be received in an own dedicated time-slot, i. e., $n \geq 4$. Conform to orthogonal time-sharing, only 1 DoF is achievable. ■

In the following section, we design a set of delay-based channels that can utilize the scheme in Theorem 1 to perfectly align interference by propagation delay.

IV. Node Placement in Euclidean Space

For the sake of simplicity, we assume that propagation delay is proportional and normalized to the Euclidean distance between each user. This assumption also occurs in [5], [9], [10]. We neglect further wireless effects as, e. g., multi-path propagation, path loss and fading and consider a line-of-sight environment.

For notational reasons, we define the following indexing for the 4 users $\mathrm{Tx}_1, \mathrm{Tx}_2, \mathrm{Rx}_1$ and Rx_2 in the X-channel:

$$\mathrm{Tx}_1 \mathrel{\hat=} 1,\ \mathrm{Tx}_2 \mathrel{\hat=} 2,\ \mathrm{Rx}_1 \mathrel{\hat=} 3,\ \mathrm{Rx}_2 \mathrel{\hat=} 4. \tag{17}$$

The Euclidean distances between each user are denoted by a symmetric *dissimilarity matrix* $\boldsymbol{\Delta} = (\delta_{ji})_{1\leq j,i\leq 4}$ with the entries $\delta_{ji} \in \mathbb{N}_{>0}$ and a zero diagonal. We specify the relationship between the propagation delay and the Euclidean distance by $d_{ji} = x^{\delta_{ji}}$. The dissimilarity matrix can be decomposed into four blocks:

$$\boldsymbol{\Delta} = \begin{pmatrix} \boldsymbol{B} & \boldsymbol{C}^{\mathsf{T}} \\ \boldsymbol{C} & \boldsymbol{A} \end{pmatrix}. \tag{18}$$

Thus, the elements of matrix $\boldsymbol{C} = (\delta_{ji})_{3\leq j\leq 4, 1\leq i\leq 2}$ correspond to the delay offset exponents of the elements in matrix $\boldsymbol{D}$ defined in Section II.

Note that the condition $\det(\boldsymbol{D}) \not\equiv 0 \bmod(x^3 - 1)$ from Theorem 1 leads to an equivalent condition for matrix $\boldsymbol{C}$:

$$\delta_{31} + \delta_{42} \not\equiv \delta_{41} + \delta_{32} \pmod 3. \tag{19}$$

Matrix $\boldsymbol{B} = (\delta_{ji})_{1\leq j,i\leq 2}$ only describes the distances between the transmitters, and matrix $\boldsymbol{A} = (\delta_{ji})_{3\leq j,i\leq 4}$ only the distances between the receivers, respectively. We may set the variables $\delta_{12} = \delta_{21} = b \in \mathbb{R}^+$ and $\delta_{34} = \delta_{43} = a \in \mathbb{R}^+$ due to symmetry and $\delta_{ii} = 0$, for $i = 1, \ldots, 4$, and obtain:

$$\boldsymbol{B} = b(\boldsymbol{1}_{2\times 2} - \boldsymbol{I}_2),$$
$$\boldsymbol{A} = a(\boldsymbol{1}_{2\times 2} - \boldsymbol{I}_2).$$

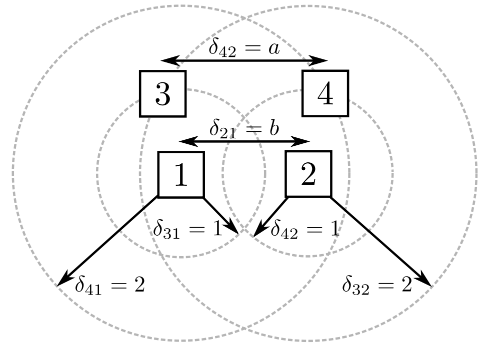

Fig. 4. A two-dimensional node placement solution is shown for a given matrix $\boldsymbol{C}$ with $\delta_{31} = 1$, $\delta_{32} = 2$, $\delta_{41} = 2$ and $\delta_{42} = 1$. The condition (19) is fulfilled, parameter b satisfies (20) and a satisfies (21).

Node placement terms a procedure of how to place user-*nodes* in Euclidean space enabling (Cyclic) IA by propagation delay. Here, we consider the problem as an *Euclidean embedding* of the dissimilarity matrix $\boldsymbol{\Delta}$ according to [9], [10] with fixed entries in the matrix $\boldsymbol{C}$ and variable entries $a, b \in \mathbb{R}^+$ in $\boldsymbol{B}, \boldsymbol{A}$, respectively. To derive the sought distances b and a, the receivers j and k, $j \neq k \in \{3, 4\}$, must be positioned s. t. all the given distances in $\boldsymbol{C}$ are fulfilled.

The objective of Euclidean embedding is to find a solution of *placement vectors* $\boldsymbol{x}_i \in \mathbb{R}^m$ for all users $i = 1, \ldots, 4$ within the lowest number of dimensions m yet satisfying the fixed distances in $\boldsymbol{\Delta}$. In our case it is desirable to find a practically most relevant solution in only $m \leq 3$ dimensions for a spatial arrangement. Fig. 3 depicts an exemplary solution of node placement for Cyclic IA by delay on the X-channel in one dimension. Another exemplary node placement in two dimensions is depicted in Fig. 4.

There is a feasible solution in two dimensions, if both transmitters satisfy all of the following (four) triangle inequalities with $j, k \in \{3, 4\}$ and $i, l \in \{1, 2\}$:

$$0 < \|\delta_{ji} - \delta_{jl}\|_2 \leq b \leq \|\delta_{ji}\|_2 + \|\delta_{jl}\|_2, \tag{20}$$
$$0 < \|\delta_{ji} - \delta_{ki}\|_2 \leq a \leq \|\delta_{ji}\|_2 + \|\delta_{ki}\|_2. \tag{21}$$

These inequalities must be non-zero because neither the transmitters Tx_1, Tx_2 nor the receivers Rx_3, Rx_4 may overlap at the same point in the Euclidean space. If the lower and upper bounds support a solution in $b, a \in \mathbb{R}$, then a 2-dimensional node placement exists.

The solution is derived from elementary geometry: W. l. o. g. we can position Tx_i on a reference point $\boldsymbol{0}_2$. To position Tx_l, $i \neq l$, some b satisfying condition (20) can be fixed on a straight line that originates in point $\boldsymbol{0}_2$. Then the valid positions for receivers Rx_j and Rx_k, $j \neq k$, can be determined easily. Valid positions for Rx_j are the intersecting points of the two circles $\mathcal{O}_{ji}$ and $\mathcal{O}_{jl}$ with $j \neq k \in \{3, 4\}$, $i \neq l \in \{1, 2\}$:

$$\begin{aligned} \mathcal{O}_{ji}: &\quad x^2 + y^2 &= \delta_{ji}^2, \\ \mathcal{O}_{jl}: &\quad (x-b)^2 + y^2 &= \delta_{jl}^2. \end{aligned}$$

The valid positions for Rx_k yield accordingly:

$$\begin{aligned} \mathcal{O}_{ki}: &\quad x^2 + y^2 &= \delta_{ki}^2, \\ \mathcal{O}_{kl}: &\quad (x-b)^2 + y^2 &= \delta_{kl}^2. \end{aligned}$$

Condition (21) is satisfied by construction, due to the fact that the circles $\mathcal{O}_{ji}$, $\mathcal{O}_{ki}$ are concentric around node i and the circles $\mathcal{O}_{jl}$, $\mathcal{O}_{kl}$ concentric around node l.

The resulting placement vectors are computed as:

$$\boldsymbol{x}_i = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \boldsymbol{x}_l = \begin{pmatrix} b \\ 0 \end{pmatrix},$$

$$\boldsymbol{x}_j = \begin{pmatrix} \frac{\delta_{j1}^2-\delta_{j2}^2+b^2}{2b} \\ \pm\sqrt{\alpha_j} \end{pmatrix}, \quad \boldsymbol{x}_k = \begin{pmatrix} \frac{\delta_{k1}^2-\delta_{k2}^2+b^2}{2b} \\ \pm\sqrt{\alpha_k} \end{pmatrix},$$

with the indices $j \neq k \in \{3,4\}$ and $i \neq l \in \{1,2\}$, and the discriminants $\alpha_3 = \delta_{31}^2 - \frac{\delta_{31}^2-\delta_{32}^2+b^2}{2b}$, $\alpha_4 = \delta_{41}^2 - \frac{\delta_{41}^2-\delta_{42}^2+b^2}{2b}$.

If both discriminants α_3, α_4 are greater than zero, then a two-dimensional solution exists. If $\alpha_3 = \alpha_4 = 0$, the solution is one-dimensional. A negative discriminant states that there is no feasible solution for node placement. The remaining distance a between the receivers is computed by:

$$a = \|\boldsymbol{x}_j - \boldsymbol{x}_k\|_2.$$

with $j \neq k \in \{3,4\}$. Moreover, such a positioning of users is invariant to rotation and translation.

A dual solution is to fix receiver Rx_j at point $\mathbf{0}_2$ and Rx_k at point $\boldsymbol{x}_k = (a,0)^\mathsf{T}$ satisfying condition (21) and finding the intersection points of the corresponding two circle pairs around Rx_j and Rx_k.

An extension to $m = 3$ dimensions would include an additional z-coordinate and the computation of two intersection circles for the four intersecting spheres. The 3-dimensional solution is also a rotational body around the connecting line δ_{21} of the 2-dimensional solution and again invariant to rotation and translation.

V. The K-User Interference Channel

In the following, an analogously defined delay-based interference channel with K user-pairs is considered. The model is depicted in Fig. 5. We assume that there is a number of $M = K$ independent messages W_i dedicated to be conveyed pair-wise from transmitter Tx_i to receiver Rx_i with indices $i \in \mathcal{K} := \{1,\dots,K\}$.

The delay matrix for this channel is defined between the transmitters Tx_i and the receivers Rx_j as $\boldsymbol{D} = (d_{ji})_{1\le i,j\le K}$. The delays d_{ji} are in $\mathcal{D} = \{x^k | k \in \mathbb{N}_0\}$ as in the case of the X-channel. This channel is also *fully-connected* as in [1].

The polynomial $v_i(x)$ contains the message W_i for the dedicated Rx_i with the parameters $p_i \in \mathbb{N}_0$ and $i \in \mathcal{K}$:

$$v_i(x) = W_i x^{p_i}. \tag{22}$$

The input vector of the transmitted polynomials $v_i(x)$ is denoted by $\boldsymbol{v} = (v_1(x),\dots,v_K(x))$. The transfer function of the delay-based interference channel yields the received vector $\boldsymbol{r} = (r_1(x),\dots,r_K(x))$ and is analogous to (2):

$$\boldsymbol{r}^\mathsf{T} = \boldsymbol{D}\boldsymbol{v}^\mathsf{T} \bmod (x^n - 1).$$

The received polynomials of Rx_j with indices $j \in \mathcal{K}$ are:

$$r_j(x) = \textstyle\sum_{i=1}^K d_{ji} W_i x^{p_i} \bmod (x^n - 1).$$

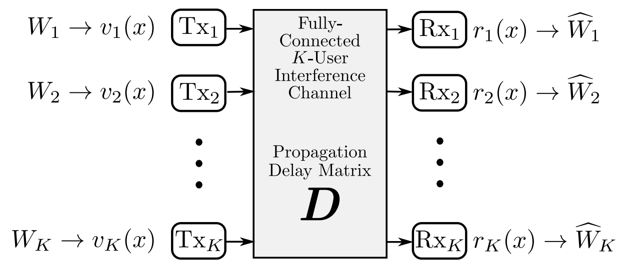

Fig. 5. The fully-connected delay-based interference channel with K user-pairs and $M = K$ messages $W_1,\dots,W_K$, s. t. there is one message between each pair of transmitters $\mathrm{Tx}_1,\dots,\mathrm{Tx}_K$ and receivers $\mathrm{Rx}_1,\dots,\mathrm{Rx}_K$ for propagation delay matrix $\boldsymbol{D}$.

The set of K encoding functions e_i and K decoding functions f_i is defined for each $i \in \mathcal{K}$ as:

$$\begin{aligned} e_i : \quad & W_i \to v_i(x), \\ f_i : \quad & r_i(x) \to \widehat{W}_i. \end{aligned}$$

The task of Cyclic IA on the given K-user interference channel is to convey and decode the $M = K$ dedicated messages $W_1,\dots,W_K$ interference-free in a period of n time-slots. The scheme is optimal in the sense of Cyclic IA if the number of time-slots n is minimal and still feasible.

In contrast to the delay-based X-channel, neither a *multiple-access interference* nor an *intra-user interference condition* is needed for the K-user interference channel, since there is only one message W_i per user-pair i anyway. Thus, only the following *inter-user interference conditions* are to be considered for $j \neq i \in \mathcal{K}$:

$$d_{jj} x^{p_j} \not\equiv d_{ji} x^{p_i} \bmod (x^n - 1). \tag{23}$$

We count $K(K-1)$ of these separability conditions in total for K users, i. e., a single receiver perceives $K-1$ inter-user interference signals from undesired transmitters.

In the given case of K user-pairs, *perfect IA* means to align all $K-1$ interfering signals received at Rx_j into a single time-slot for all $j \in \mathcal{K}$:

$$d_{ji} x^{p_i} \equiv d_{jk} x^{p_k} \bmod (x^n - 1), \tag{24}$$

with pair-wise distinct $i,j,k \in \mathcal{K}$.

For notational convenience, we define 2×2 submatrices of $\boldsymbol{D}$ denoted as $\boldsymbol{D}_{j,k,i}$ if the following structure is satisfied for pair-wise distinct indices $i,j,k \in \mathcal{K}$:

$$\boldsymbol{D}_{j,k,i} = \begin{pmatrix} d_{jj} & d_{ji} \\ d_{kj} & d_{ki} \end{pmatrix}. \tag{25}$$

Cyclic IA on the K-user interference channel can achieve $\frac{K}{2}$ DoF if all the conditions of the following Theorem 3 hold.

Theorem 3. *A perfect Cyclic IA scheme for the K-user interference channel exists, if the three conditions:*

- $\det(\boldsymbol{D}_{j,k,i}) \not\equiv 0 \bmod (x^n - 1)$,
- $d_{ji} d_{kj} d_{ik} \equiv d_{ij} d_{jk} d_{ki} \bmod (x^n - 1)$,
- *and* $n \geq 2$,

hold with pair-wise distinct $i,j,k \in \mathcal{K}$. Then, Cyclic IA by propagation delay achieves $\frac{K}{2}$ DoF on the delay-based K-user interference channel.

Proof:

(a) *Necessity of* $d_{ji}d_{kj}d_{ik} \equiv d_{ij}d_{jk}d_{ki} \bmod(x^n-1)$, $n \in \mathbb{N}$:
Let $i, j, k \in \mathcal{K}$ be pair-wise distinct. By relabeling the indices in (24) to $i \to j$, $j \to k$ and $k \to i$, we obtain (26). And by relabeling the indices in (24) to $i \to k$, $j \to i$ and $k \to j$, we obtain (27), resp.:

$$(24): d_{ji}x^{p_i} \equiv d_{jk}x^{p_k} \bmod(x^n-1),$$
$$d_{kj}x^{p_j} \equiv d_{ki}x^{p_i} \bmod(x^n-1), \tag{26}$$
$$d_{ik}x^{p_k} \equiv d_{ij}x^{p_j} \bmod(x^n-1). \tag{27}$$

These three equivalences are solvable, if and only if the following holds:

$$d_{ji}d_{kj}d_{ik} \equiv d_{jk}d_{ij}d_{ki} \bmod(x^n-1). \tag{28}$$

Otherwise, perfect IA cannot be applied.

(b) *Necessity of* $\det(\boldsymbol{D}_{j,k,i}) \not\equiv 0$, $n \in \mathbb{N}$:
Inserting (26) into the separability condition of the inter-user interference (23) yields:

$$d_{jj}d_{ki}d_{kj}^{-1}x^{p_i} \not\equiv d_{ji}x^{p_i} \bmod(x^n-1)$$
$$\Rightarrow 1 \not\equiv d_{ji}d_{kj}d_{ki}^{-1}d_{jj}^{-1} \bmod(x^n-1)$$
$$\Rightarrow 0 \not\equiv \det(\boldsymbol{D}_{j,k,i}) \bmod(x^n-1), \tag{29}$$

for pair-wise distinct $i, j, k \in \mathcal{K}$. If $\det(\boldsymbol{D}_{j,k,i}) \equiv 0$ holds for any pair-wise distinct $i, j, k \in \mathcal{K}$, the separability conditions cannot be fulfilled for perfect IA.

(c) *Necessity of* $n > 1$ time-slot:
Only $n = 1$ time-slot would preclude any separation of desired and interfering messages necessary for (23).

(d) *Sufficiency of* $d_{ji}d_{kj}d_{ik} \equiv d_{ij}d_{jk}d_{ki} \bmod(x^n-1)$ *and* $\det(\boldsymbol{D}_{j,k,i}) \not\equiv 0$ *and* $n = 2$ *for pair-wise distinct* $i, j, k \in \mathcal{K}$ *to achieve* $\frac{K}{2}$ *DoF*:
Firstly, we can assume the valid propagation delay matrix $\boldsymbol{D} = x^1\boldsymbol{I}_K + (\boldsymbol{1}_{K\times K} - \boldsymbol{I}_K)x^2$ as also considered in [1, Appx. I]. The condition (28) holds since any non-diagonal entry is x^2, i.e., $d_{ji}d_{kj}d_{ik} \equiv d_{ij}d_{jk}d_{ki} \equiv x^2x^2x^2 \equiv 1 \bmod(x^2-1)$. The condition (29) also holds since $\det(\boldsymbol{D}_{j,k,i}) \equiv x^3(x^1-1) \not\equiv 0 \bmod(x^2-1)$.
There are further valid delay matrices $\boldsymbol{D}$: If all entries of a row in the given $\boldsymbol{D}$ are right-shifted by $x^m, m \in \mathbb{N}$, the two conditions still hold, e.g., for a shifted row j we obtain $(d_{ji}x^m)d_{kj} \equiv d_{ki}(d_{jj}x^m) \bmod(x^2-1)$.
W.l.o.g., we can fix p_1 and determine all other p_i with $i \neq 1 \in \mathcal{K}$ by applying the perfect IA condition in (24):

$$x^{p_i} \equiv d_{ji}^{-1}d_{j1}x^{p_1} \bmod(x^2-1).$$

Altogether, K messages can be conveyed interference-free in $n = 2$ time-slots, i.e., $\frac{K}{2}$ DoF are achieved. This also achieves an upper bound as we will show in Section VI, Lemma 5.

■

A corresponding node placement for this special case of delay matrix $\boldsymbol{D}$ is investigated in [5], [9]. However, a general solution for node placement of K users with any valid delay matrix $\boldsymbol{D}$ exceeds the scope of the present paper.

VI. Upper Bounds on the Degrees of Freedom in Delay-Based Interference Channels

In the following, we provide an upper bound on the DoF to prove the optimality of Theorems 1 and 3 in terms of Cyclic IA. The given upper bound is analogous to [4, Thm. 1], albeit applied to the delay-based interference channel.

We assume a fully-connected multi-user interference channel with K_{T} transmitters and K_{R} receivers. This allows us to derive the upper bounds for quite a general set of delay-based interference channels. The number of desired messages for any receiver-transmitter pair $(\mathrm{Rx}_j, \mathrm{Tx}_i)$ within n time-slots is expressed by a *messaging matrix* $\boldsymbol{M} = (m_{ji})_{1\le j\le K_{\mathrm{R}}, 1\le i\le K_{\mathrm{T}}}$ with $m_{ji} \in \mathbb{N}_0$. The total number of desired messages M is simply computed by adding all elements in the messaging matrix $\boldsymbol{M}$. We denote the j-th row vector of $\boldsymbol{M}$ by $\boldsymbol{\mu}_j$ s.t. $\boldsymbol{M} = (\boldsymbol{\mu}_1^{\mathsf{T}}, \ldots, \boldsymbol{\mu}_{K_{\mathrm{R}}}^{\mathsf{T}})^{\mathsf{T}}$ holds:

$$\boldsymbol{\mu}_j = (m_{j1}, \ldots, m_{jK_{\mathrm{T}}}).$$

The number of desired messages to be conveyed from all transmitters to Rx_j corresponds to summing up the entries in the j-th row vector:

$$\|\boldsymbol{\mu}_j\|_1 = \textstyle\sum_{i=1}^{K_{\mathrm{R}}} m_{ji}.$$

The messages transmitted from Tx_i are given by the i-th column vector $\boldsymbol{\nu}_i$ of $\boldsymbol{M}$ s.t. $\boldsymbol{M} = (\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_{K_{\mathrm{T}}})$ holds:

$$\boldsymbol{\nu}_i = (m_{1i}, \ldots, m_{K_{\mathrm{R}}i}).$$

The total number of messages from Tx_i is computed by the 1-norm of the i-th column vector $\boldsymbol{v}_i$, respectively:

$$\|\boldsymbol{\nu}_i\|_1 = \textstyle\sum_{j=1}^{K_{\mathrm{R}}} m_{ji}.$$

In general the *multiple-access interference condition* demands separability of all $\|\boldsymbol{\mu_j}\|_1$ desired messages, s.t. these messages are received interference-free at Rx_j. Furthermore, the *intra-user interference condition* demands separability of all $\|\boldsymbol{\nu_i}\|_1$ messages transmitted from Tx_i. For a pair $(\mathrm{Rx}_j, \mathrm{Tx}_i)$, at least $\|\boldsymbol{\mu_j}\|_1 + \|\boldsymbol{\nu_i}\|_1 - m_{ji}$ time-slots ensure the two given separability conditions. Since entry m_{ji} appears twice when adding the 1-norm of row and column vectors, it must be subtracted once.

To further ensure the *inter-user interference condition*, the maximum of the sum $\|\boldsymbol{\mu_j}\|_1 + \|\boldsymbol{\nu_i}\|_1 - m_{ji}$ over all $i, j \in \mathcal{K}$ provides the minimal feasible n. Altogether, the minimal number of n time-slots in a fixed messaging matrix $\boldsymbol{M}$ is lower bounded by:

$$n \ge \max_{\boldsymbol{\nu}_j, \boldsymbol{\mu}_i, m_{ji}} (\|\boldsymbol{\mu}_j\|_1 + \|\boldsymbol{\nu}_i\|_1 - m_{ji})$$
$$= \max_{m_{ji}} \Big(\textstyle\sum_{i=1}^{K_{\mathrm{T}}} m_{ji} + \sum_{j=1}^{K_{\mathrm{R}}} m_{ji} - m_{ji}\Big), \tag{30}$$

with $j \in \{1, \ldots, K_{\mathrm{R}}\}$ and $i \in \{1, \ldots, K_{\mathrm{T}}\}$.

The DoF defined in (3) are the number of messages per in one period of n time-slots and hence upper bounded by:

$$\mathrm{DoF} \le \frac{\sum_{j=1}^{K_{\mathrm{R}}} \sum_{i=1}^{K_{\mathrm{T}}} m_{ji}}{\max\limits_{m_{ji}} \big(\sum_{i=1}^{K_{\mathrm{T}}} m_{ji} + \sum_{j=1}^{K_{\mathrm{R}}} m_{ji} - m_{ji}\big)}. \tag{31}$$

We do not prove whether the upper bound is tight in the general case. Nonetheless, the following two Lemmas 4 and 5 yield tight upper bounds to Theorem 1 and 3, respectively.

Lemma 4. *The upper bound on the DoF of the delay-based 2-user X-channel is $\frac{4}{3}$ DoF and tight.*

Proof:
The messaging matrix of an X-channel with $K_{\mathrm{R}} = K_{\mathrm{T}} = 2$ is defined as the 2×2 matrix of ones:

$$\boldsymbol{M} = (\boldsymbol{\mu}_1^{\mathsf{T}}, \boldsymbol{\mu}_2^{\mathsf{T}})^{\mathsf{T}} = \mathbf{1}_{2\times 2}.$$

The total number of desired messages in $\boldsymbol{M}$ is $M = 4$. The 1-norm of each row vector and of each column vector yields:

$$\mu := \|\boldsymbol{\mu}_1\|_1 = \|\boldsymbol{\mu}_2\|_1 = 2,$$
$$\nu := \|\boldsymbol{\nu}_1\|_1 = \|\boldsymbol{\nu}_2\|_1 = 2.$$

Furthermore, $\eta := m_{ji} = 1$ holds for all $i, j \in \mathcal{K}$. The minimal number of time-slots is lower bounded by:

$$n \geq \mu + \nu - \eta = 3.$$

The upper bound of $\frac{4}{3}$ DoF is tight in this case since the achievability shown in Theorem 1. ■

Lemma 5. *The upper bound on the DoF of the delay-based K-user interference channel is $\frac{K}{2}$ DoF and tight.*

Proof:
The messaging matrix of a K-user interference channel with $K_{\mathrm{T}} = K_{\mathrm{R}} = K$ is defined as the identity matrix:

$$\boldsymbol{M} = (\boldsymbol{\mu}_1^{\mathsf{T}}, \ldots, \boldsymbol{\mu}_K^{\mathsf{T}})^{\mathsf{T}} = \boldsymbol{I}_K.$$

The total number of transmitted messages is $M = K$. We obtain the following 1-norms for each row and column vector, resp.:

$$\mu := \|\boldsymbol{\mu}_1\|_1 = \cdots = \|\boldsymbol{\mu}_K\|_1 = 1,$$
$$\nu := \|\boldsymbol{\nu}_1\|_1 = \cdots = \|\boldsymbol{\nu}_K\|_1 = 1.$$

The $\eta := m_{ji} = 0$ is minimal for any $j \neq i \in \mathcal{K}$. The minimal number of time-slots is lower bounded by:

$$n \geq \nu + \mu - \eta = 2.$$

The resulting upper bound of $\frac{K}{2}$ DoF is tight in this case since the achievability is shown in Theorem 3. ■

VII. Discussion

The main ingredient to enable the proposed Cyclic IA scheme is the cyclicality of the discrete, normalized propagation delays in the n-periodic channel access. Being aware of the fact that the consideration of such constrained propagation delays seems quite artificial w. r. t. contemporary wireless channel models, we believe that our proposed method might be transferable to other models which include similar comparable cyclic properties aside from propagation delay. The basic model is capable to describe multi-user interference and to support the concept of interference alignment. We show the practicability to formulate simple conditions that have also been observed in more established channel models and to proof optimality criteria. As it has been done similarly for the *linear deterministic* channel model [6], we conjecture that schemes developed on the Cyclic IA might be translated to practical models in a subsequent step. We intended to reveal a less extensive solution that focuses on basic traits of Cyclic IA and that permits an easier mathematical generalization.

The recent works [13], [14] also consider a related cyclic setup for submarine communication systems applying a scheduling approach. In a work on line-of-sight channels [7], certain cyclic effects are observed for IA to some extend. But the considered approach relies heavily on computationally extensive graph-theoretic approaches.

The given figures and delays do not specify the absolute scale of node placement. The idea could be implemented in a very large scale, e. g., in deep space, where long propagation delays are present and large time-slots can be used as mentioned in [12]. In that case, the synchronization of the delayed time-slots should be maintained sufficiently static and accurately predictable, even if many users are involved. Vice versa, it is possible to consider a very small scale of the time-slots and the node placement [5] s. t. node placement becomes a placement problem of antennas. However, the synchronization would surely be very challenging.

VIII. Conclusions

In this work, we formalized the communication model for Interference Alignment by propagation delay utilizing cyclic right-shifts of polynomials as known from cyclic codes. The given model is elaborated for a delay-based X-channel and a delay-based K-user interference channel.

A set of defined separability conditions ensure the decodability of all messages involved in the considered communication scenarios. These separability conditions discern multiple-access, inter- and intra-user interference. This classification of interference facilitates the derivation of sufficient and necessary conditions for the feasibility and for the optimal solutions of the *Cyclic Interference Alignment* scheme as presented in Theorem 1 and Theorem 3.

Beyond that, it is shown under certain conditions, that the users can be positioned in Euclidean space by a *node placement* scheme such that Cyclic Interference Alignment by propagation delay is valid in two dimensions for the given 2-user X-channel.

We also remark that the Cyclic Interference Alignment scheme is not limited to the considered delay-based scenarios and presume that it might be portable to any channel model that involves comparable cyclic properties.

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REFERENCES

[1] V. Cadambe and S. Jafar, “Interference alignment and degrees of freedom of the K-user interference channel,” *IEEE Trans. Inform. Theory*, vol. 54, no. 8, pp. 3425–3441, Aug. 2008.

[2] M. A. Maddah-Ali, A. S. Motahari, and A. K. Khandani, “Signaling over MIMO multi-base systems: Combination of multi-access and broadcast schemes,” in *Proc. IEEE Int. Symp. Inform. Theory (ISIT)*, Seattle, WA, USA, Jul. 2006, pp. 2104–2108.

[3] S. Jafar, *Interference Alignment: A New Look at Signal Dimensions in a Communication Network*. Foundations and Trends in Communications and Information Theory, 2011, vol. 7, no. 1.

[4] V. Cadambe and S. Jafar, “Interference alignment and the degrees of freedom of wireless X networks,” *IEEE Trans. Inform. Theory*, vol. 55, no. 9, pp. 3893–3908, Sep. 2009.

[5] ——, “Degrees of freedom of wireless networks - what a difference delay makes,” in *Proc. of Asilomar Conf. on Signals, Systems and Computers (ACSSC)*, Pacific Grove, CA, USA, Nov. 2007, pp. 133–137.

[6] V. Cadambe, S. Jafar, and S. Shamai (Shitz), “Interference alignment on the deterministic channel and application to fully connected Gaussian interference networks,” *IEEE Trans. Inform. Theory*, vol. 55, no. 1, pp. 269–274, Jan. 2009.

[7] L. Grokop, D. Tse, and R. D. Yates, “Interference alignment for line-of-sight channels,” *IEEE Trans. Inform. Theory*, vol. 57, no. 9, pp. 5820–5839, Sep. 2011.

[8] V. Cadambe and S. Jafar, “Can 100 speakers talk for 30-minutes each in one room within one hour and with zero interference to each other’s audience?” in *Proc. of 45th Annual Allerton Conference on Comm., Control and Comp.*, Monticello, IL, USA, Sep. 2007, pp. 1141–1148.

[9] R. Mathar and M. Zivkovic, “How to position n transmitter-receiver pairs in $n - 1$ dimensions such that each can use half of the channel with zero interference from the others,” in *Proc. IEEE Global Telecomm. Conf. (GLOBECOM)*, Honolulu, HI, USA, Dec. 2009, pp. 1–4.

[10] R. Mathar and G. Böcherer, “On spatial patterns of transmitter-receiver pairs that allow for interference alignment by delay,” in *Proc. 3rd Int. Conf. on Sig. Proc. and Comm. Sys. (ICSPCS)*, Omaha, NE, USA, Sep. 2009, pp. 1–5.

[11] H. Zhou and T. Ratnarajah, “A novel interference draining scheme for cognitive radio based on interference alignment,” in *Proc. IEEE Intern. Dynamic Spectrum Access Networks (DySpAN)*, Singapore, Apr. 2010.

[12] F. L. Blasco, F. Rosetto, and G. Bauch, “Time interference alignment via delay offset for long delay networks,” in *Proc. IEEE Global Telecomm. Conf. (GLOBECOM)*, Houston, TX, USA, 2011.

[13] M. Chitre, M. Motani, and S. Shahabudeen, “A scheduling algorithm for wireless networks with large propagation delays,” in *Proc. IEEE OCEANS’ 2010*, Sydney, Australia, May 2010, pp. 1–5.

[14] ——, “Throughput of networks with large propagation delays,” *IEEE Journal of Oceanic Engineering*, May 2012, accepted.