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Exercise 2

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Problem 1. *(Properties of expectation and covariance)* Two independent random vectors $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)^{\mathrm{T}}$ and $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)^{\mathrm{T}}$ with $n \in \mathbb{N}$ are given. Furthermore, c_X, c_Y, $\boldsymbol{A}$ and $\boldsymbol{b}$ are fixed quantities of adequate dimensions. Prove the following identities:

a) (Scale and shift properties) $\mathrm{E}(\boldsymbol{AX} + \boldsymbol{b}) = \boldsymbol{A}\,\mathrm{E}(\boldsymbol{X}) + \boldsymbol{b}$,

b) (linearity) $\mathrm{E}(c_X\boldsymbol{X} + c_Y\boldsymbol{Y}) = c_X\,\mathrm{E}(\boldsymbol{X}) + c_Y\,\mathrm{E}(\boldsymbol{Y})$,

c) (independency) $\mathrm{E}(\boldsymbol{XY}) = \mathrm{E}(\boldsymbol{X})\,\mathrm{E}(\boldsymbol{Y})$,

d) $\mathrm{Cov}(\boldsymbol{AX} + \boldsymbol{b}) = \boldsymbol{A}\,\mathrm{Cov}(\boldsymbol{X})\,\boldsymbol{A}^{\mathrm{H}}$,

e) $\mathrm{Cov}(c_X\boldsymbol{X} + c_Y\boldsymbol{Y}) = |c_X|^2\,\mathrm{Cov}(\boldsymbol{X}) + |c_Y|^2\,\mathrm{Cov}(\boldsymbol{Y})$.

Problem 2. *(Mean and covariance of normal distribution)* Let $\mathbf{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $n \in \mathbb{N}$ be given. Show the identities $\mathrm{E}(\mathbf{X}) = \boldsymbol{\mu}$ and $\mathrm{Cov}(\mathbf{X}) = \boldsymbol{\Sigma}$.

Problem 3. *(Higher moments)* Let $\boldsymbol{X} \sim \mathcal{N}_n(\mathbf{0}, \boldsymbol{\Sigma})$ with $n \in \mathbb{N}$ be given. Use the Isserlis' Theorem to calculate the higher moments $\mathrm{E}(X_1X_2X_3X_4X_5)$, $\mathrm{E}(X_1X_2X_3X_4)$ and $\mathrm{E}(X_1^2X_5^4)$.

Isserlis' Theorem: If $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ is a zero mean multivariate normal random vector with covariance $\boldsymbol{\Sigma}$, then

$$\mathrm{E}(X_1X_2\cdots X_m) = \begin{cases} 0, & \text{if } m \le n \text{ is odd,} \\ \displaystyle\sum_{\boldsymbol{\pi}} \prod_{\substack{i,j \\ i \ne j}} \mathrm{E}(X_iX_j), & \text{if } m \le n \text{ is even.} \end{cases}$$

The sum is performed over all permutations $\boldsymbol{\pi}$ for partitioning the sequence $X_1, X_2, \ldots, X_m$ in pairs of two random variables.