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Exercise 3

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Problem 1. *(Matrix Derivative)* Let $\mathbf{A}$ and $\mathbf{X}$ be $n \times n$ matrix and $\mathbf{x}, \mathbf{y}$ be $n-$dimensional vectors. Prove the following statements:

a) $\dfrac{\partial \mathbf{y}^T \mathbf{x}}{\partial \mathbf{x}} = \mathbf{y}$

b) $\dfrac{\partial \mathbf{x}^T \mathbf{A} \mathbf{x}}{\partial \mathbf{x}} = (\mathbf{A} + \mathbf{A}^T)\mathbf{x}$

c) $\dfrac{\partial}{\partial \mathbf{X}} \mathrm{tr}(\mathbf{X}\mathbf{A}) = \mathbf{A}^T$

d) $\dfrac{\partial}{\partial \mathbf{X}} \mathrm{tr}(\mathbf{X}^T \mathbf{A} \mathbf{X}) = (\mathbf{A}^T + \mathbf{A})\mathbf{X}$

e) $\dfrac{\partial}{\partial \mathbf{X}} \|\mathbf{X}\|_F^2 = 2\mathbf{X}$

f) $\dfrac{\partial}{\partial \mathbf{X}} \det(\mathbf{X}) = \det(\mathbf{X})(\mathbf{X}^{-1})^T$

g) $\dfrac{\partial}{\partial \mathbf{X}} \log|\det(\mathbf{X})| = (\mathbf{X}^{-1})^T$

Problem 2. *(Unbiased Covariance Estimator)* If $\mathbf{X}_1, \ldots, \mathbf{X}_n$ are random i.i.d. draws from a multivariate distribution, prove that the sample mean $\overline{\mathbf{X}}$ and sample covariance matrix $\mathbf{S}_n$ are unbiased estimators of expected value $\mathbb{E}(\mathbf{X})$ and covariance matrix $\mathbf{\Sigma} = \mathrm{Cov}(\mathbf{X})$ of the multivariate distribution. The sample mean and covariance are defined as follows:

$$\overline{\mathbf{X}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{X}_i, \quad \mathbf{S}_n = \frac{1}{n-1}\sum_{i=1}^{n} (\mathbf{X}_i - \overline{\mathbf{X}})(\mathbf{X}_i - \overline{\mathbf{X}})^T.$$

Problem 3. *(PCA in 3-dimensional space)* Consider four samples in $\mathbb{R}^3$ given as follows:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} \mathbf{x}_3 = \begin{bmatrix} -4 \\ 2 \\ 2 \end{bmatrix} \mathbf{x}_4 = \begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix}.$$

a) Find the sample mean and the sample covariance matrix.

b) Using PCA, find the best orthogonal projection matrix **Q** for presenting the data in two dimensional space. Explain each step.

c) Show that the image of **Q** is the plane $x + y + z = 0$.