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Exercise 6

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Problem 1. *(Matrix power)* Show that for any matrix $\boldsymbol{A}$ with eigenvalues λ_i and corresponding eigenvectors $\boldsymbol{x}_i$, the eigenvalues μ_i and eigenvectors $\boldsymbol{y}_i$ of $\boldsymbol{A}^k$, $k \in \mathbb{N}$, are determined by λ_i^k and $\boldsymbol{x}_i$, respectively.

Problem 2. *(Dominant eigenvector and eigenvalue estimation)* Show that for any matrix $\boldsymbol{A}$ with eigenvalues $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$ and corresponding eigenvectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$, the vector $\boldsymbol{y}_k = \boldsymbol{A}^k \boldsymbol{y}_0$, $k \in \mathbb{N}$, converges to the direction of the dominant eigenvector of $\boldsymbol{A}$, i.e., $\lim_{k \mapsto \infty} \boldsymbol{y}_k \mapsto c\boldsymbol{x}_1\, , c \in \mathbb{R}$.

Show that $\frac{\boldsymbol{y}_k^{\mathrm{T}} \boldsymbol{y}_{k+1}}{\|\boldsymbol{y}_k\|_2^2}$ converges to the dominant eigenvalue λ_1.

Problem 3. *(Properties of stochastic matrices)* Let $\boldsymbol{A}$ be a stochastic matrix. Show the following statements:

a) Exchanging two rows or two columns of $\boldsymbol{A}$ will again result in a stochastic matrix $\boldsymbol{B}$ with the same structure. In other words, any permutation of rows or columns of a right, left, or double stochastic matrix will lead to a right, left, or double stochastic matrix, respectively.

b) If $\boldsymbol{A}$ is block-diagonal (and consequently square) then the number of unitary eigenvalues is greater than the number of blocks. Specify the corresponding eigenvectors.

c) Using both above statements helps to find the dominant eigenvalues and eigenvectors of any stochastic matrix $\boldsymbol{A}$ assuming that $\boldsymbol{A}$ can be written in terms of the permutated matrix $\boldsymbol{B}$ as $\boldsymbol{B} = \boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^{-1}$ given a permuatation matrix $\boldsymbol{P}$. Determine the eigenvalues and corresponding eigenvectors of $\boldsymbol{A}$ in terms of eigenvalues and eigenvectors of $\boldsymbol{B}$ for a given permutation matrix $\boldsymbol{P}$. How is the structure of the dominant eigenvectors of $\boldsymbol{A}$ when $\boldsymbol{B}$ is block-diagonal?

Definition of stochastic matrices: A stochastic matrix is a matrix with nonnegative real entries representing probabilities. We say a matrix is right (left) stochastic if in each row (column) the sum of all entries is equal to one. Right and left stochastic matrices are often called row and column stochastic matrices, respectively. If a matrix is both right and left stochastic, we say the matrix is double stochastic.

Definition of permutation matrices: A permutation matrix is a matrix with only a single nonzero entry in each row and column. Usually all nonzero entries are equal to one.

Problem 4. *(Multiple Unit Eigenvalues of Stochastic Matrix)* Suppose thet $G = (V, E, \mathbf{W})$ is a weighted graph with the symmetric weight matrix $\mathbf{W}$. Suppose that the transition matrix of a random walk on this graph is denoted by $\mathbf{M}$.

a) Prove that $\mathbf{M}$ has multiple eigenvalues eqaul to one if and only if the graph is disconnected.

b) If the underlying graph G is connected, prove that $\mathbf{M}$ has an eigenvalue eqaul to -1 if and only if the graph is bipartite.

Problem 5. *(Diffusion Distance)* Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be some points in $\mathbb{R}^p$, and the garph $(V, E, \mathbf{W})$ is constructed based on those points using a kernel function. Transition probability matrix is constructed accordingly. Suppose that the diffusion map of a vertex v_i is given by $\boldsymbol{\phi}_t(v_i)$. For any pair of nodes v_i and v_j in the graph, prove:

$$\|\boldsymbol{\phi}_t(v_i) - \boldsymbol{\phi}_t(v_j)\|^2 = \sum_{l=1}^{n} \frac{1}{\deg(l)} \Big(\mathbb{P}(X_t = l | X_0 = i) - \mathbb{P}(X_t = l | X_0 = j)\Big)^2.$$