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Exercise 6
- Proposed Solution -

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Solution of Problem 1

The identity $\boldsymbol{A}\boldsymbol{x}_i = \boldsymbol{x}_i\lambda_i$ holds by definition. Multiplying both sides by $\boldsymbol{A}$ leads to $\boldsymbol{A}\boldsymbol{A}\boldsymbol{x}_i = \boldsymbol{A}\boldsymbol{x}_i\lambda_i = \boldsymbol{x}_i\lambda_i^2$ which shows the equalities $\mu_i = \lambda_i^2$ and $\boldsymbol{y}_i = \boldsymbol{x}_i$ for $k = 2$. Repeating this procedure k-times in total proves the statement.

Solution of Problem 2

We can represent any vector $\boldsymbol{y}_0$ by $\sum_{i=1}^n c_i\boldsymbol{x}_i$ with proper weights $c_i \in \mathbb{R}$. Incorporation of this combination into the definition of $\boldsymbol{y}_k$ leads to

$$\begin{aligned}\boldsymbol{y}_k &= \boldsymbol{A}^k\boldsymbol{y}_0 = \boldsymbol{A}^k\sum_{i=1}^n c_i\boldsymbol{x}_i = \sum_{i=1}^n c_i\boldsymbol{A}^k\boldsymbol{x}_i \\ &= \sum_{i=1}^n c_i\lambda_i^k\boldsymbol{x}_i = \lambda_1^k\left(c_1\boldsymbol{x}_1 + \sum_{i=2}^n c_i\left[\frac{\lambda_i}{\lambda_1}\right]^k\boldsymbol{x}_i\right).\end{aligned}$$

Since $|\lambda_1| > |\lambda_i|$ for all $i > 1$, the ratio $\frac{\lambda_i}{\lambda_1}$ tends to zero for $k \mapsto \infty$, i.e., $\lim_{k\mapsto\infty}\frac{y_k}{\lambda_1^k} \mapsto c_1\boldsymbol{x}_1$.

To show the limit of $\frac{\boldsymbol{y}_k^{\mathrm{T}}\boldsymbol{y}_{k+1}}{\|\boldsymbol{y}_k\|_2^2}$ we use the above relation for $\boldsymbol{y}_k$ to conclude

$$\begin{aligned}\lim_{k\mapsto\infty}\frac{\boldsymbol{y}_k^{\mathrm{T}}\boldsymbol{y}_{k+1}}{\|\boldsymbol{y}_k\|_2^2} &= \lim_{k\mapsto\infty}\frac{\lambda_1^k\left(c_1\boldsymbol{x}_1 + \sum_{i=2}^n c_i\left[\frac{\lambda_i}{\lambda_1}\right]^k\boldsymbol{x}_i\right)^{\mathrm{T}}\lambda_1^{k+1}\left(c_1\boldsymbol{x}_1 + \sum_{i=2}^n c_i\left[\frac{\lambda_i}{\lambda_1}\right]^{k+1}\boldsymbol{x}_i\right)}{\left\|\lambda_1^k\left(c_1\boldsymbol{x}_1 + \sum_{i=2}^n c_i\left[\frac{\lambda_i}{\lambda_1}\right]^k\boldsymbol{x}_i\right)\right\|_2^2} \\ &= \lambda_1\lim_{k\mapsto\infty}\frac{\left(c_1\boldsymbol{x}_1 + \sum_{i=2}^n c_i\left[\frac{\lambda_i}{\lambda_1}\right]^k\boldsymbol{x}_i\right)^{\mathrm{T}}\left(c_1\boldsymbol{x}_1 + \sum_{i=2}^n c_i\left[\frac{\lambda_i}{\lambda_1}\right]^{k+1}\boldsymbol{x}_i\right)}{\left\|c_1\boldsymbol{x}_1 + \sum_{i=2}^n c_i\left[\frac{\lambda_i}{\lambda_1}\right]^k\boldsymbol{x}_i\right\|_2^2} \\ &= \lambda_1\lim_{k\mapsto\infty}\frac{\left(c_1\boldsymbol{x}_1 + \sum_{i=2}^n c_i\left[\frac{\lambda_i}{\lambda_1}\right]^k\boldsymbol{x}_i\right)^{\mathrm{T}}\left(c_1\boldsymbol{x}_1 + \sum_{i=2}^n c_i\left[\frac{\lambda_i}{\lambda_1}\right]^k\boldsymbol{x}_i\right)}{\left|c_1\boldsymbol{x}_1 + \sum_{i=2}^n c_i\left[\frac{\lambda_i}{\lambda_1}\right]^k\boldsymbol{x}_i\right|^2} = \lambda_1.\end{aligned}$$

Solution of Problem 3

a) Exchanging two rows or two columns of $\boldsymbol{A}$ will not change the nonnegativity of its entries. It also does not change the row or the column sums. In this way, the structure of a stochastic matrix will remain after permutation of its rows or columns.

b) Consider a block-diagonal and stochastic matrix $\boldsymbol{A}$ of size $n \times n$ with diagonal entries $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_m$. The size of the sub-matrix $\boldsymbol{A}_i$ is assumed to be $n_i \times n_i$ for all $1 \leq i \leq m$. Note that the sum $\sum_{i=1}^{m} n_i$ is equal to n. Since the square matrices $\boldsymbol{A}_i$ must also be stochastic matrices, for each of them there exist a eigenvalue equal to one and a corresponding eigenvector whose entries are all one. We can simply extend the size of these eigenvectors from n_i to n by including zero-entries before and/or after the ones. In this way, we can in total generate m eigenvectors with unitary eigenvalues by the aid of the block structure. Another unitary eigenvalue with an eigenvector whose entries are all one is known for the matrix $\boldsymbol{A}$ itself. Hence we obtain $m+1$ unitary eigenvalues.

c) The eigenvalues and eigenvectors of $\boldsymbol{B}$ are given by definition $\boldsymbol{B}\boldsymbol{x}_i = \boldsymbol{x}_i\lambda_i$. Substituting $\boldsymbol{B}$ and multiplying both sides of the last equation by $\boldsymbol{P}^{-1}$ leads to $\boldsymbol{A}\boldsymbol{P}^{-1}\boldsymbol{x}_i = \boldsymbol{P}^{-1}\boldsymbol{x}_i\lambda_i$ which shows that the eigenvalues μ_i and eigenvectors $\boldsymbol{y}_i$ of $\boldsymbol{A}$ are equal to λ_i and $\boldsymbol{P}^{-1}\boldsymbol{x}_i$, respectively. In case that $\boldsymbol{B}$ is block-diagonal, the dominant eigenvectors of $\boldsymbol{A}$ have only one and zero entries.

Solution of Problem 4

a) ***Forward direction***:

If the graph is disconnected, then the vertex set V can be partitioned into two sets A and B such that no edge exists between A and B. In the transition matrix $\mathbf{M}$, non-zero entries appear only on the entries inside $A \times A$ and $B \times B$. Let $\mathbf{M}_{C,D}$ is constructed as the submatrix of $\mathbf{M}$ by choosing rows from the set C and columns from the set D. Then $\mathbf{M}_{A,A}$ and $\mathbf{M}_{B,B}$ are both transition matrices on their own. Then $\boldsymbol{\chi}_A$ and $\boldsymbol{\chi}_B$ are both eigenvectors of those matrices with eigenvalues equal to one, where $\boldsymbol{\chi}_A = (\chi_A(i))_{1 \leq i \leq n}$, with $\chi_A(x) = 0$ if $x \notin A$ and $\chi_A(x) = 1$ if $x \in A$. Therefore there are at least two eigenvalues equal to one in this case.

Reverse direction:

Suppose that there is more than one eigenvalue equal to one.

However, it is known that $|\lambda_k| \leq 1$ for all eigenvalues of $\mathbf{M}$.

Let $\mathbf{m} = (m_1, \ldots, m_n)^T$ be the eigenvector corresponding to $\lambda_2 = 1$. If $|m_l| = \max_{1 \leq j \leq n} |m_j|$, we have:

$$|\lambda_2| \leq \sum_{j=1}^{n} M_{lj} \frac{|m_j|}{|m_l|} \leq \sum_{j=1}^{n} M_{lj} = 1.$$

The equality obtains if $\frac{|m_j|}{|m_l|} = 1$ for those j where $M_{lj} \neq 0$ and m_j's have those same sign. We can assume that $m_l = 1$. Define the the set A as:

$$A = \{k : m_k = 1\}.$$

Since the first eigenvector is $\mathbf{1}_n$, the second eigenvector should be different and hence $A \neq \{1, \ldots, n\}$ and $A^c \neq \emptyset$. For each $k \in A$, we have:

$$\sum_{j=1}^{n} M_{kj} m_j = 1.$$

Note that $\sum_{j=1}^n M_{kj} = 1$ and $m_j < 1$ for $j \in A^c$. Therefore to have the equality, again $m_j = 1$ whenever $M_{kj} \neq 0$. In other words, $M_{ij} = 0$ for $j \in A^c$ and $i \in A$. Since $M_{ij} = \frac{w_{ij}}{\deg(i)}$, $M_{ji} = 0$ for $j \in A^c$ and $i \in A$. In other words there is no edge between the nodes of A and A^c. Hence the graph is disconnected.

Similar argument can be used by looking at m_k, $k \in A^c$. Since $M_{ki} = 0$ for $i \in A$, then $\sum_{j \in A^c} M_{kj} m_j = m_k$. Taking a k maximum absolute value entry and applying the same argument, another set B can be constructed having $m_j = m_k$ for $j \in B$ (this time we do not have $m_k = 1$). The set B represents another connected component of the graph. The process can be continued by removing the previous connected components from the graph and analysing again the remaining graph.

b) ***Forward direction***:

If the graph is bipartite, then the vertex set V can be partitioned into two sets A and B such that no edge exists inside A and inside B but only between them. In the transition matrix $\mathbf{M}$, non-zero entries appear only on the entries inside $A \times B$ and $B \times A$. Without loss of generality assume that $A = \{1, \dots, l\}$ and $B = \{l+1, \dots, n\}$. Then:

$$\mathbf{M} \begin{bmatrix} \mathbf{1}_A \\ \mathbf{0}_B \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{A\times A} & \mathbf{M}_{A\times B} \\ \mathbf{M}_{B\times A} & \mathbf{0}_{B\times B} \end{bmatrix} \begin{bmatrix} \mathbf{1}_A \\ \mathbf{0}_B \end{bmatrix} = \begin{bmatrix} \mathbf{0}_A \\ \mathbf{1}_B \end{bmatrix}$$

$$\mathbf{M} \begin{bmatrix} \mathbf{0}_A \\ \mathbf{1}_B \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{A\times A} & \mathbf{M}_{A\times B} \\ \mathbf{M}_{B\times A} & \mathbf{0}_{B\times B} \end{bmatrix} \begin{bmatrix} \mathbf{0}_A \\ \mathbf{1}_B \end{bmatrix} = \begin{bmatrix} \mathbf{1}_A \\ \mathbf{0}_B \end{bmatrix}$$

That implies:

$$\mathbf{M} \begin{bmatrix} \mathbf{1}_A \\ -\mathbf{1}_B \end{bmatrix} = \begin{bmatrix} -\mathbf{1}_A \\ \mathbf{1}_B \end{bmatrix}.$$

Therefore -1 is an eigenvalue.

Reverse direction:

Suppose that $\lambda_2 = -1$ is an eigenvalue. If $|m_l| = \max_{1 \le j \le n} |m_j|$, we have:

$$|\lambda| \le \sum_{j=1}^n M_{lj} \frac{|m_j|}{|m_l|} \le \sum_{j=1}^n M_{lj} = 1.$$

Since $\lambda = -1$, the equality obtains if $m_j = -m_l$ for those j where $M_{lj} \neq 0$. We can assume that $m_l = 1$. Define the the sets A_{-1}, A_1, A_0 as:

$$A_{-1} = \{k : m_k = -1\}, A_1 = \{k : m_k = 1\}, A_0 = \{k : |m_k| \neq 1\}.$$

For each $k \in A_{-1}$, we have:

$$\sum_{j=1}^n M_{kj} m_j = 1.$$

This means that if $k \in A_{-1}$, then $m_j = 1$ for $M_{kj} \neq 0$. In other words, if $m_j \neq 1$ then $M_{kj} = 0$, which is:

$$M_{kj} = 0 \text{ for } k \in A_{-1}, j \in A_1^c.$$

In other words no edge exists between the vertices inside A_{-1}, also there is no edge from A_{-1} to A_0. The edges from A_{-1} go only to A_1.

Similarly for each $k \in A_1$, we have:

$$\sum_{j=1}^{n} M_{kj} m_j = -1.$$

This means that if $k \in A_1$, then $m_j = -1$ for $M_{kj} \neq 0$. In other words, if $m_j \neq -1$ then $M_{kj} = 0$, which is:

$$M_{kj} = 0 \text{ for } k \in A_1, j \in A_{-1}^c.$$

In other words no edge exists between the vertices inside A_1, also there is no edge from A_1 to A_0. The edges from A_1 go only to A_{-1}.

This means that $A_1 \cup A_{-1}$ is a component which is bipartite and it is disconnected from A_0. Since it is assumed that the graph is connected, then $A_0 = \emptyset$.

Solution of Problem 5

First of all, see that:

$$\begin{aligned}
&\sum_{l=1}^{n} \frac{1}{\deg(l)} \Big(\mathbb{P}(X_t = l | X_0 = i) - \mathbb{P}(X_t = l | X_0 = j)\Big)^2 \\
&\quad = \sum_{l=1}^{n} \frac{1}{\deg(l)} \left(\sum_{k=1}^{n} \lambda_k^t \phi_{k,i} \psi_{k,l} - \sum_{k=1}^{n} \lambda_k^t \phi_{k,i} \psi_{k,l} \right)^2 = \sum_{l=1}^{n} \frac{1}{\deg(l)} \left(\sum_{k=1}^{n} \lambda_k^t (\phi_{k,i} - \phi_{k,i}) \psi_{k,l} \right)^2 \\
&\quad = \sum_{l=1}^{n} \left(\sum_{k=1}^{n} \lambda_k^t (\phi_{k,i} - \phi_{k,i}) \frac{\psi_{k,l}}{\sqrt{\deg(l)}} \right)^2 = \left\| \sum_{k=1}^{n} \lambda_k^t (\phi_{k,i} - \phi_{k,i}) \mathbf{D}^{-1/2} \boldsymbol{\psi}_k \right\|^2
\end{aligned}$$

Note that $\mathbf{D}^{-1/2}\mathbf{\Psi}$ is equal to $\mathbf{V}^T$, the eigenvalue matrix in spectral decomposition of $\mathbf{S}$. Therefore $\mathbf{D}^{-1/2}\boldsymbol{\psi}_k$'s are orthonormal, and we have:

$$\left\| \sum_{k=1}^{n} \lambda_k^t (\phi_{k,i} - \phi_{k,i}) \mathbf{D}^{-1/2} \boldsymbol{\psi}_k \right\|^2 = \sum_{k=1}^{n} \lambda_k^t (\phi_{k,i} - \phi_{k,i})^2 = \|\boldsymbol{\phi}_t(v_i) - \boldsymbol{\phi}_t(v_j)\|^2.$$