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Exercise 2

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Problem 1. *(Matrix Loewner Ordering Properties)* Let $\mathbf{V}$ and $\mathbf{W}$ be two $n \times n$ non-negative definite matrices, such that $\mathbf{V} = (v_{ij}) \preceq \mathbf{W} = (w_{ij})$, with the eignevalues as:

- $\lambda_1(\mathbf{V}) \geq \cdots \geq \lambda_n(\mathbf{V})$,
- $\lambda_1(\mathbf{W}) \geq \cdots \geq \lambda_n(\mathbf{W})$

Prove the following statements.

a) $\lambda_i(\mathbf{V}) \leq \lambda_i(\mathbf{W})$, for $i = 1, \dots, n$

b) $v_{ii} \leq w_{ii}$, for $i = 1, \dots, n$

c) $v_{ii} + v_{jj} - 2v_{ij} \leq w_{ii} + w_{jj} - 2w_{ij}$

d) $\operatorname{tr}(\mathbf{V}) \leq \operatorname{tr}(\mathbf{W})$

e) $\det(\mathbf{V}) \leq \det(\mathbf{W})$

Problem 2. *(Properties of Isometries)* Let $\mathbf{A}$ be $n \times n$ be an isometry on $\mathbb{R}^n$.

a) Prove that $\mathbf{A}$ is full rank.

b) Find the singular values of $\mathbf{A}$.

c) Find the Frobenius norm of $\mathbf{A}$.

Problem 3. *(Distribution of eigenvalues)* Use Gerschgorin's Theorem to find the smallest regions in which the eigenvalues of the matrix $\boldsymbol{A}$ are concentrated. Is $\boldsymbol{A}$ positive definite? Determine the smallest interval $[\lambda_{\min}, \lambda_{\max}]$ in which the real part of the eigenvalues are distributed.

$$\boldsymbol{A} = \begin{pmatrix} 10 & 0.1 & 1 & 0.9 & 0 \\ 0.2 & 9 & 0.2 & 0.2 & 0.2 \\ 0.3 & -0.1 & 5+i & 0 & 0.1 \\ 0 & 0.6 & 0.1 & 6 & -0.3 \\ 0.3 & -0.3 & 0.1 & 0 & 1 \end{pmatrix}$$

Gerschgorin's Theorem: Let $\boldsymbol{A} \in \mathbb{C}^{n\times n}$, with entries a_{ij}, be given. For $i, j \in \{1, \dots, n\}$ let $R_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$ and $C_j = \sum_{\substack{i=1 \\ i\neq j}}^{n} |a_{ij}|$ be the sum of the absolute values of the non-diagonal entries. Then every eigenvalue of $\boldsymbol{A}$ lies within at least one of the discs centered at a_{ii} with radius $\min\{R_i, C_i\}$.

Note that if one of the discs is disjoint from the others then it contains exactly one eigenvalue. If the union of m discs is disjoint from the union of the other $n - m$ discs then the former union contains exactly m and the latter $n - m$ eigenvalues of $\boldsymbol{A}$.