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Exercise 3

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Problem 1. *(Properties of expectation and covariance)* Two independent random vectors $\boldsymbol{X} = (X_1, X_2, \dots, X_n)^{\mathrm{T}}$ and $\boldsymbol{Y} = (Y_1, Y_2, \dots, Y_n)^{\mathrm{T}}$ with $n \in \mathbb{N}$ are given. Furthermore, c_X, c_Y, $\boldsymbol{A}$ and $\boldsymbol{b}$ are fixed quantities of adequate dimensions. Prove the following identities:

a) (Scale and shift properties) $\mathrm{E}(\boldsymbol{AX} + \boldsymbol{b}) = \boldsymbol{A}\,\mathrm{E}(\boldsymbol{X}) + \boldsymbol{b}$,

b) (Linearity) $\mathrm{E}(c_X \boldsymbol{X} + c_Y \boldsymbol{Y}) = c_X\,\mathrm{E}(\boldsymbol{X}) + c_Y\,\mathrm{E}(\boldsymbol{Y})$,

c) (Independence) $\mathrm{E}(\boldsymbol{X}^T \boldsymbol{Y}) = \mathrm{E}(\boldsymbol{X})^T\,\mathrm{E}(\boldsymbol{Y})$,

d) $\mathrm{Cov}(\boldsymbol{AX} + \boldsymbol{b}) = \boldsymbol{A}\,\mathrm{Cov}(\boldsymbol{X})\,\boldsymbol{A}^{\mathrm{H}}$,

e) $\mathrm{Cov}(c_X \boldsymbol{X} + c_Y \boldsymbol{Y}) = |c_X|^2\,\mathrm{Cov}(\boldsymbol{X}) + |c_Y|^2\,\mathrm{Cov}(\boldsymbol{Y})$.

Problem 2. *(Bivariate Distribution)*

Suppose that $(Y_1, Y_2) \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$

Then obtain an expression in terms of $\mu_1, \mu_2, \sigma_1, \sigma_2, \rho \in \mathbb{R}$ for the following distributions.

a) The joint distribution $f_{Y_1,Y_2}(y_1, y_2)$.

b) The distribution of Y_1 and the distribution of Y_2.

c) The conditional density $f_{Y_1}(y_1|y_2)$.

Problem 3. *(Maximum Likelihood Estimation)*

Suppose that the random variable X is absolutely continuous with the density $f_X(x)$ where

$$f_X(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

where $\lambda > 0$. Assume that we want to use Maximum Likelihood Estimation (MLE) to estimate λ from n independent observations of X, denoted as $\mathbf{x} = (x_1, \dots, x_n)$.

a) Write down the log-likelihood function.

b) What is the MLE of the parameter λ?