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Exercise 5

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Problem 1. *(Centering Matrix)*

For a set of column vectors $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)} \in \mathbb{R}^k$ and the centering matrix $\mathbf{E}_k = \mathbf{I}_k - \frac{1}{k}\mathbf{1}_k\mathbf{1}_k^{\mathrm{T}} \in \mathbb{R}^{k\times k}$, let $\mathbf{X} \in \mathbb{R}^{n\times k}$ be $\mathbf{X} = [\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}]^{\mathrm{T}}$ and $\overline{x}^{(j)} = \frac{1}{k}\sum_{i=1}^{k} x_i^{(j)}$, where $x_i^{(j)}$ is the i-th entry of $\mathbf{x}^{(i)}$.

a) Show that $\mathbf{E}_k\mathbf{x}^{(j)} = \mathbf{x}^{(j)} - \overline{x}^{(j)}\mathbf{1}_k$.

b) Show that the (i, j)-th entry of $\mathbf{E}_k\mathbf{X}^{\mathrm{T}}$ is given by $x_i^{(j)} - \overline{x}^{(j)}$.

c) Show that $\sum_{i=1}^{k}\left(\mathbf{E}_k\mathbf{X}^{\mathrm{T}}\right)_{i,j} = 0$ for any $j \in \{1, 2, \dots, n\}$.

Problem 2. *(Characterization of Euclidean Distance Matrices)*

a) Show that if $\mathbf{D}(\mathbf{X}) \in \mathbb{R}^{n\times n}$ is a distance matrix, then show that:

$$-\frac{1}{2}\mathbf{D}^{(2)}(\mathbf{X}) = \mathbf{X}\mathbf{X}^T - \mathbf{1}_n\hat{\mathbf{x}}^T - \hat{\mathbf{x}}\mathbf{1}^T$$

where $\hat{\mathbf{x}} = \frac{1}{2}[\mathbf{x}_1^T\mathbf{x}_1, \dots, \mathbf{x}_n^T\mathbf{x}_n]^T$.

b) Consider $-\frac{1}{2}\mathbf{E}_n\boldsymbol{\Delta}^{(2)}\mathbf{E}_n$, which is non-negative definite and $\mathrm{rk}(-\frac{1}{2}\mathbf{E}_n\boldsymbol{\Delta}^{(2)}\mathbf{E}_n) \le k$, then there exists $n \times k$ matrix $\mathbf{X}$ such that

$$-\frac{1}{2}\mathbf{E}_n\boldsymbol{\Delta}^{(2)}\mathbf{E}_n = \mathbf{X}\mathbf{X}^T, \text{ and } \mathbf{X}^T\mathbf{E}_n = \mathbf{X}^T.$$

c) A matrix with zero diagonal elements is called hollow matrix. Prove that if $\mathbf{A}$ is a symmetric hollow matrix, then $\mathbf{A} = 0$ if and only if $\mathbf{E}_n\mathbf{A}\mathbf{E}_n = 0$.

Problem 3. *(Spike model II)*

Show that if $\beta > \sqrt{\gamma}$ then $(1+\beta)(1+\frac{\gamma}{\beta}) > (1+\sqrt{\gamma})^2$.