Prof. Dr. Rudolf Mathar, Dr. Arash Behboodi, Emilio Balda

Exercise 11

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Problem 1. *(Dual Problem for Linear and Quadratic Programming)*

a) Consider the linear programming problem defined as follows:

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \preceq \mathbf{b} \end{aligned}$$

Find the dual problem.

b) Suppose that $\mathbf{B}$ is positive definite matrix and consider the following quadratic programming:

$$\begin{aligned} \min \quad & \mathbf{x}^T \mathbf{B} \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \preceq \mathbf{b}. \end{aligned}$$

Find the dual problem.

c) For $p \geq 1$, consider the following norm minimization problem:

$$\begin{aligned} \min \quad & \|\mathbf{x}\|_p \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}. \end{aligned}$$

Find the dual problem.

Problem 2. *(Support Vector Machine for Non-separable Classes)* Consider the following SVM optimization problem for a non-separable dataset:

$$\begin{aligned} \min_{\mathbf{a},b,\boldsymbol{\xi}} \quad & \frac{1}{2}\|\mathbf{a}\|^2 + c\sum_{i=1}^{n} \xi_n \\ \text{s.t.} \quad & y_i(\mathbf{a}^T \mathbf{x}_i + b) \geq 1 - \xi_i \quad i = 1, \dots, n \\ & \xi_i \geq 0 \quad i = 1, \dots, n. \end{aligned} \tag{1}$$

a) Find the dual problem of this optimization problem.

b) Suppose that support vectors and optimal $\mathbf{a}^\star$ are given. Find the optimal $\mathbf{b}^\star$.

Problem 3. Support Vector Machines:

a) Suppose that a training dataset is composed of vectors $\mathbf{x}_i \in \mathbb{R}^3$ belonging to two classes. The class membership is indicated by the labels $y_i \in \{-1, +1\}$. Suppose that the dataset is separable. A Support vector machine is used to find the maximum-margin hyperplane $\mathbf{a}^{\mathrm{T}}\mathbf{x} + b = 0$. The primal optimization problem gives the optimal $\mathbf{a}^\star$ as $\begin{pmatrix} 1 & 3 & 0 \end{pmatrix}^{\mathrm{T}}$. Two support vectors with different labels are given as :

$$\mathbf{x}_1^{\mathrm{T}} = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}, \qquad \mathbf{x}_2^{\mathrm{T}} = \begin{pmatrix} -1 & -1 & -1 \end{pmatrix}$$

Find the optimal value $b^\star$.

Consider another training dataset that is non-separable. The following dual problem is solved for a support vector machine

$$\max_{\boldsymbol{\lambda}} \quad \sum_{i=1}^{6} \lambda_i - \frac{1}{2} \sum_{i,j} y_i y_j \lambda_i \lambda_j \mathbf{x}_i^T \mathbf{x}_j$$

$$\text{s.t.} \quad 0 \leq \lambda_i \leq 5 \qquad \text{and} \qquad \sum_{i=1}^{6} \lambda_i y_i = 0.$$

The dataset with the outputs of the optimization problem are given in the following table.

Data	Label	Solution	Data	Label	Solution
$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$y_1 = -1$	$\lambda_1^\star = 0$	$\mathbf{x}_4 = \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix}$	$y_4 = 1$	$\lambda_4^\star = 4.73$
$\mathbf{x}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$	$y_2 = -1$	$\lambda_2^\star = 0.67$	$\mathbf{x}_5 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$	$y_5 = 1$	$\lambda_5^\star = 0.94$
$\mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$y_3 = -1$	$\lambda_3^\star = 5$	$\mathbf{x}_6 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$	$y_6 = 1$	$\lambda_1^\star = 0$

b) Determine the support vectors.

c) Find the maximum-margin hyperplane by finding $\mathbf{a}^\star$ and $b^\star$.