Prof. Dr. Rudolf Mathar, Dr. Arash Behboodi, Emilio Balda

Exercise 2
- Proposed Solution -

Friday, November 3, 2017

Solution of Problem 1

a) Since $\mathbf{W} \succeq \mathbf{V}$, $\mathbf{W} - \mathbf{V}$ is non-negative definite. Therefore $\mathbf{x}^T(\mathbf{W} - \mathbf{V})\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$, which means:

$$\mathbf{x}^T\mathbf{W}\mathbf{x} \geq \mathbf{x}^T\mathbf{V}\mathbf{x}.$$

Using Courant-Fischer theorem, it is known that:

$$\max_{S:\dim(S)=k} \min_{\mathbf{x}\in S; \|x\|_2=1} \mathbf{x}^T\mathbf{W}\mathbf{x} = \lambda_k(\mathbf{W}).$$

and

$$\max_{S:\dim(S)=k} \min_{x\in S; \|x\|_2=1} \mathbf{x}^T\mathbf{V}\mathbf{x} = \lambda_k(\mathbf{V}).$$

However $\mathbf{x}^T\mathbf{W}\mathbf{x} \geq \mathbf{x}^T\mathbf{V}\mathbf{x}$ implies that $\lambda_k(\mathbf{W}) \geq \lambda_k(\mathbf{V})$.

b) Since $\mathbf{W} \succeq \mathbf{V}$, $\mathbf{W} - \mathbf{V}$ is non-negative definite. Therefore $\mathbf{x}^T(\mathbf{W} - \mathbf{V})\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Choose $\mathbf{x} = \mathbf{e}_i$ where $\mathbf{e}_i$ is ith canonical basis with all zero elements except the ith element equal to one. Namely $\mathbf{e}_i(j) = 0$ for $j \neq i$ and $\mathbf{e}_i(i) = 1$. For example:

$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Therefore $\mathbf{e}_i^T(\mathbf{W} - \mathbf{V})\mathbf{e}_i = w_{ii} - v_{ii}$ and since $\mathbf{W} - \mathbf{V} \succeq 0$, $w_{ii} - v_{ii} \geq 0$. $v_{ii} \leq w_{ii}$, for $i = 1, \ldots, n$

c) Similar to the previous problem, choose the vector $\mathbf{e}_{ij}$ such that $\mathbf{e}_{ij}(k) = 0$ for $j \neq i, j$ and $\mathbf{e}_{ij}(i) = 1$ and $\mathbf{e}_{ij}(j) = -1$. For example:

$$\mathbf{e}_{23} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{bmatrix}$$

Since $\mathbf{W} - \mathbf{V} \succeq 0$, $\mathbf{e}_{ij}^T(\mathbf{W} - \mathbf{V})\mathbf{e}_{ij} \geq 0$, but:

$$(\mathbf{W} - \mathbf{V})\mathbf{e}_{ij} = \begin{bmatrix} (w_{1i} - v_{1i}) - (w_{1j} - v_{1j}) \\ (w_{2i} - v_{2i}) - (w_{2j} - v_{2j}) \\ \vdots \\ (w_{ni} - v_{ni}) - (w_{nj} - v_{nj}) \end{bmatrix}$$

and

$$\begin{aligned}\mathbf{e}_{ij}^T(\mathbf{W}-\mathbf{V})\mathbf{e}_{ij} &= [(w_{ii}-v_{ii})-(w_{ij}-v_{ij})]-[(w_{ji}-v_{ji})-(w_{jj}-v_{jj})] \\ & [w_{ii}+w_{jj}-2w_{ij}]-[v_{ii}+v_{jj}-2v_{ij}]\,.\end{aligned}$$

Since $\mathbf{e}_{ij}^T(\mathbf{W}-\mathbf{V})\mathbf{e}_{ij} \geq 0$, it holds that: $v_{ii}+v_{jj}-2v_{ij} \leq w_{ii}+w_{jj}-2w_{ij}$.

d) From the second part of the exercise, $v_{ii} \leq w_{ii}$, for $i=1,\dots,n$. Therefore :

$$\operatorname{tr}(\mathbf{V}) = \sum_{i=1}^{n} v_{ii} \leq \sum_{i=1}^{n} w_{ii} = \operatorname{tr}(\mathbf{W}).$$

e) Note that $\det(\mathbf{V}) = \prod_{i=1}^n \lambda_i(\mathbf{V})$ and $\det(\mathbf{W}) = \prod_{i=1}^n \lambda_i(\mathbf{W})$. Using the first part of this exercise $\lambda_i(\mathbf{V}) \leq \lambda_i(\mathbf{W})$, for $i=1,\dots,n$. Since all eigenvalues are non-negative, it holds that $\det(\mathbf{V}) \leq \det(\mathbf{W})$.

Solution of Problem 2

(Properties of Isometries) Let $\mathbf{A}$ be $n \times n$ be an isometry on $\mathbb{R}^n$.

a) If $\mathbf{A}$ is not full rank, then there is non-zero $\mathbf{x} \in \mathbb{R}^n$ for which $\mathbf{Ax} = \mathbf{0}$. But in this case $\|\mathbf{Ax}\| = 0$ while isometry property implies that $\|\mathbf{Ax}\| = \|\mathbf{x}\| \neq 0$ which is a contradiction.

b) Note that the singular values of $\mathbf{A}$ are square root of eigenvalues of $\mathbf{A}^T\mathbf{A}$. The matrix $\mathbf{A}^T\mathbf{A}$ is positive definite and has n non-zero eigenvalues. Suppose that λ and $\mathbf{v} \neq 0$ are an eigenvalue-eigenvector pair. We have:

$$\mathbf{A}^T\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \implies \mathbf{v}^T\mathbf{A}^T\mathbf{A}\mathbf{v} = \lambda\|\mathbf{v}\|^2.$$

From isometry property of the matrix $\mathbf{A}$, we have:

$$\mathbf{v} \implies \mathbf{v}^T\mathbf{A}^T\mathbf{A}\mathbf{v} = \|\mathbf{v}\|^2 \implies \|\mathbf{v}\|^2 = \lambda\|\mathbf{v}\|^2 \implies \lambda = 1.$$

Therefore all eigenvalues are equal to one and hence the singular values of $\mathbf{A}$ are equal to one.

c) The Frobenius norm of $\mathbf{A}$ is given by:

$$\|\mathbf{A}\|_F = \sqrt{\operatorname{tr}(\mathbf{A}^T\mathbf{A})}.$$

From the previous step, all eigenvalues of $\mathbf{A}^T\mathbf{A}$ are equal to one and the trace of a matrix is equal to some of its eigenvalues. Since the matrix is full rank, it has n non-zero eigenvalues all of them equal to one. Therefore

$$\|\mathbf{A}\|_F = \sqrt{\operatorname{tr}(\mathbf{A}^T\mathbf{A})} = \sqrt{n}$$

Table 1: The centers and radii of Gerschgorin's circles

i	a_{ii}	r_i	R_i	C_i
1	10	0.8	2.0	0.8
2	9	0.8	0.8	1.1
3	5+i	0.5	0.5	1.4
4	6	1.0	1.0	1.1
5	1	0.6	0.7	0.6

Solution of Problem 3

The radii $r_i = \min\{R_i, C_i\}$ of the discs are calculated by the aid of $R_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$ and $C_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}|$, and are given in the following table. The diagonal elements of $\boldsymbol{A}$ are the centers of the discs.

From the below figure we can observe that all areas of the circles are located on the right side of the plane. But having positive eigenvalues is not sufficient for $\boldsymbol{A}$ being positive definite. Since it is not symmetric, it will not be positive definite. Furthermore, we observe the limits $\lambda_{\min} = a_{55} - r_5 = 0.4$ and $\lambda_{\max} = a_{11} + r_1 = 10.8$. Note that since the disc located at a_{55} is disjoint from the others it contains exactly one of the eigenvalues.