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Exercise 4
- Proposed Solution -

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Solution of Problem 1

a) The eigenvalues of $\mathbf{S}_n$ are solutions of $\det(\mathbf{S}_n - \mathbf{I}\lambda) = 0$. This leads to

$$\begin{vmatrix} 14-\lambda & -14 \\ -14 & 110-\lambda \end{vmatrix} = (14-\lambda)(110-\lambda) - 14^2 = \lambda^2 - 124\lambda + 1344 = (112-\lambda)(12-\lambda) = 0\,.$$

Hence, the diagonal matrix is determined by

$$\mathbf{\Lambda} = \begin{pmatrix} 112 & 0 \\ 0 & 12 \end{pmatrix}\,.$$

The eigenvectors $\mathbf{S}_n$ are solutions of $\mathbf{S}_n\mathbf{v} = \mathbf{v}\lambda$. In addition the eigenvectors should be normalized, i.e., $\|\mathbf{v}\| = 1$. We obtain

$$\begin{pmatrix} 14 & -14 \\ -14 & 110 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 112\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \Rightarrow \quad v_2 = -7v_1 \quad \text{and} \quad v_1^2 + v_2^2 = 1\,,$$

which yields the normalized eigenvector $\begin{pmatrix} \frac{1}{\sqrt{50}} & \frac{-7}{\sqrt{50}} \end{pmatrix}^{\mathrm{T}}$ for the eigenvalue 112. For the next eigenvector we only need to swap the entries of the first eigen vector and change the sign of one entry. This leads to the eigenvector $\begin{pmatrix} \frac{7}{\sqrt{50}} & \frac{1}{\sqrt{50}} \end{pmatrix}^{\mathrm{T}}$ for the eigenvalue 12. Putting the eigenvectors together we deduce the matrix

$$\mathbf{V} = \frac{1}{\sqrt{50}}\begin{pmatrix} 1 & 7 \\ -7 & 1 \end{pmatrix}\,.$$

b) The best projection matrix $\mathbf{Q}$ is determined by the first k dominant eigenvectors $\mathbf{v}_i$ as $\mathbf{Q}\sum_{i=1}^{k}\mathbf{v}_i\mathbf{v}_i^{\mathrm{T}}$, where k is the dimension of the image. For a transformation of a two-dimensional sample to a one-dimensional data (k=1), we obtain

$$\mathbf{Q} = \frac{1}{\sqrt{50}}\begin{pmatrix} 1 \\ -7 \end{pmatrix}\frac{1}{\sqrt{50}}\begin{pmatrix} 1 & -7 \end{pmatrix} = \frac{1}{50}\begin{pmatrix} 1 & -7 \\ -7 & 49 \end{pmatrix}\,.$$

c) The residuum $\frac{1}{n-1}\max_{\mathbf{Q}}\sum_{i=1}^{n}\|\mathbf{Q}\mathbf{x}_i - \mathbf{Q}\bar{\mathbf{x}}_n\|^2$ is equal to the sum $\sum_{i=1}^{k}\lambda(\mathbf{S}_n)$ of dominant eigenvalues, that is equal to 112 in the present case.

Solution of Problem 2

$$\mathbf{x}_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix}-1\\0\end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix}0\\1\end{pmatrix}, \quad \mathbf{x}_4 = \begin{pmatrix}0\\-1\end{pmatrix}$$

a) The sample covariance matrix $\mathbf{S}_n$ is given by:

$$\mathbf{S}_4 = \begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}.$$

and the spectral decomposition is trivially given. $\mathbf{V}$ can be any orthogonal matrix and $\mathbf{\Lambda}$ is the same as $\mathbf{S}_4$.

b) There is no single best projection matrix $\mathbf{Q}$; every vector $\mathbf{v}$ gives a single dimensional projection $\mathbf{v}\mathbf{v}^T$. Just two examples: $\mathbf{v} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}0\\1\end{pmatrix}$. The first projection matrix is given by

$$\mathbf{Q} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \implies \mathbf{Q}\mathbf{x}_1 = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{Q}\mathbf{x}_2 = \begin{pmatrix}-1\\0\end{pmatrix}$$

and the second projection matrix is given by

$$\mathbf{Q} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \implies \mathbf{Q}\mathbf{x}_1 = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{Q}\mathbf{x}_2 = \begin{pmatrix}0\\0\end{pmatrix}.$$

to transform the two-dimensional samples to a one-dimensional data and calculate the projection of $\mathbf{x}_1$ and $\mathbf{x}_2$.

Solution of Problem 3

a) The dominant eigenvalue λ_{dom} is visible when the ratio $\gamma_2 = \frac{p}{n_2}$ is less than β_{dom}^2. With $\beta_{\text{dom}} = \beta_2 = 0.5$ we obtain $n_{\min} = n_2 = \frac{p}{\beta_2^2} = 2000$. For this number of samples, the dominant eigenvalue of the sample covariance $\mathbf{S}_n$ tends to $(1 + \sqrt{\gamma_2})^2 = (1 + 0.5)^2 = 2.25 \gg 1.5$. The distance $\langle \boldsymbol{v}_2, \boldsymbol{v}_{\text{dom}} \rangle = \frac{1-\gamma_1/\beta_1^2}{1-\gamma_1/\beta_1}$ is equal to zero. Figure 1 shows eigenvalue distributions for this choice.

b) To see both eigenvalues the ratio $\gamma_1 = \frac{p}{n_1}$ must be less than β_1^2. With $\beta_1 = 0.2$ we obtain $n_1 = \frac{p}{\beta_1^2} = 12500$. For this number of samples, the dominant eigenvalue λ_{dom} of the sample covariance $\mathbf{S}_n$ tends to $(1 + \beta_2)(1 + \frac{\gamma_1}{\beta_2}) = 1.5 \cdot 1.08 = 1.62 \approx 1.5 = 1 + \beta_2$. The distance $\langle \boldsymbol{v}_2, \boldsymbol{v}_{\text{dom}} \rangle = \frac{1-\gamma_1/\beta_2^2}{1-\gamma_1/\beta_2}$ is equal to $0.913 \approx 1$ which shows that $\boldsymbol{v}_2$ is nearly a unit norm vector parallel to the dominant eigenvector $\boldsymbol{v}_{\text{dom}}$. Figure 2 shows eigenvalue distributions for this choice.

By enlarging n to 50000 both eigenvalues β_1 and β_2 become visible in the Marchenko-Pastur density as shown in Figure 3.

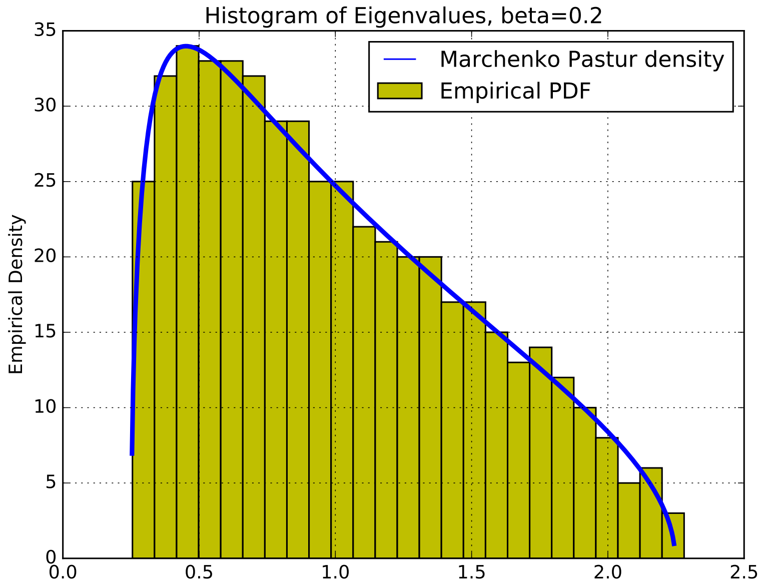

Figure 1: Eigenvalues of $\mathbf{S}_n$ for Spike model with $\beta_1 = 0.2, \beta_2 = 0.5$, $n = 2000$

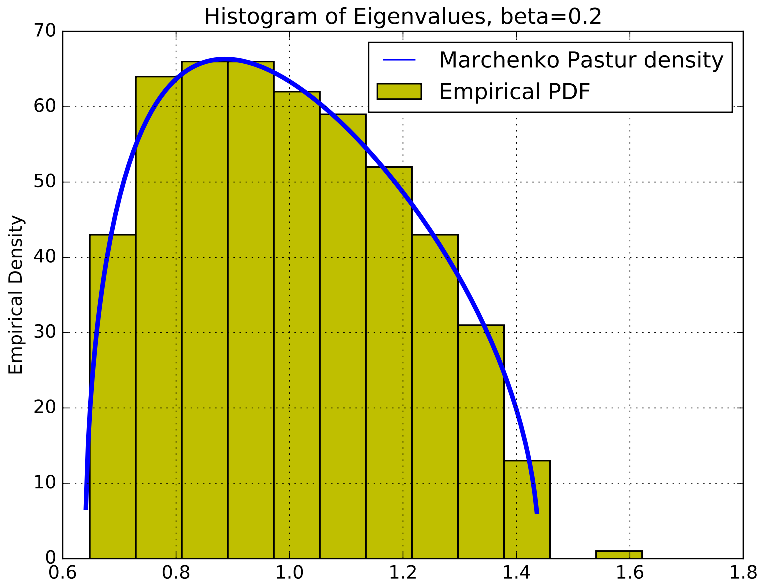

Figure 2: Eigenvalues of $\mathbf{S}_n$ for Spike model with $\beta_1 = 0.2, \beta_2 = 0.5$, $n = 12500$

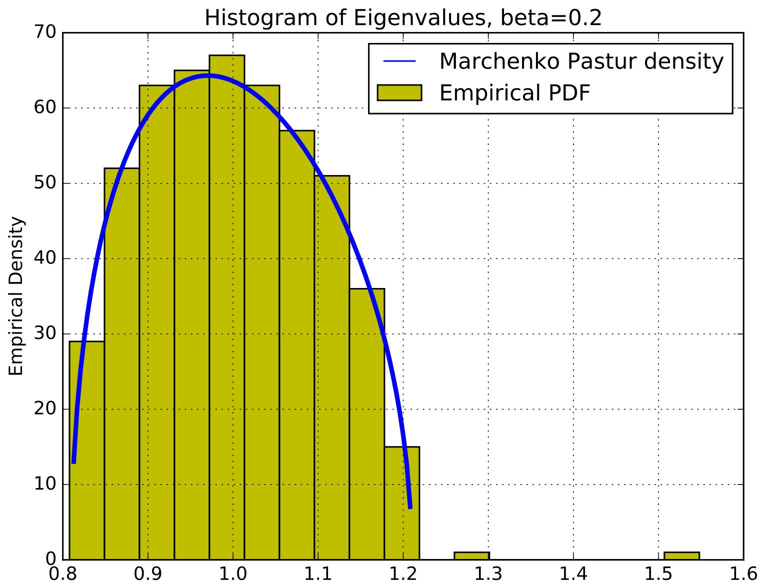

Figure 3: Eigenvalues of $\mathbf{S}_n$ for Spike model with $\beta_1 = 0.2, \beta_2 = 0.5$, $n = 50000$