Prof. Dr. Rudolf Mathar, Dr. Arash Behboodi, Emilio Balda

Exercise 5
- Proposed Solution -

Friday, November 24, 2017

Solution of Problem 1

a)

$$\mathbf{E}_k\mathbf{x}^{(j)} = \left(\mathbf{I}_k - \frac{1}{k}\mathbf{1}_k\mathbf{1}_k^{\mathrm{T}}\right)\mathbf{x}^{(j)} = \mathbf{x}^{(j)} - \frac{1}{k}\mathbf{1}_k\mathbf{1}_k^{\mathrm{T}}\mathbf{x}^{(j)} = \mathbf{x}^{(j)} - \frac{1}{k}\sum_{i=1}^{k} x_i^{(j)}\mathbf{1}_k$$
$$= \mathbf{x}^{(j)} - \overline{x}^{(j)}\mathbf{1}_k$$

b)

$$\left(\mathbf{E}_k\mathbf{X}^{\mathrm{T}}\right)_{ij} = \left[\mathbf{E}_k\mathbf{x}^{(1)}, \mathbf{E}_k\mathbf{x}^{(2)}, \dots, \mathbf{E}_k\mathbf{x}^{(n)}\right]_{ij} = \left(\mathbf{x}^{(j)} - \overline{x}^{(j)}\mathbf{1}_k\right)_i = x_i^{(j)} - \overline{x}^{(j)}$$

c)

$$\sum_{i=1}^{k}\left(\mathbf{E}_k\mathbf{X}^{\mathrm{T}}\right)_{ij} = \sum_{i=1}^{k}\left(x_i^{(j)} - \overline{x}^{(j)}\right) = \sum_{i=1}^{k} x_i^{(j)} - \sum_{i=1}^{k}\overline{x}^{(j)} = k\overline{x}^{(j)} - k\overline{x}^{(j)} = 0$$

Solution of Problem 2

a) Note that:

$$\|\mathbf{x}_i - \mathbf{x}_j\|^2 = \mathbf{x}_i^T\mathbf{x}_i + \mathbf{x}_j^T\mathbf{x}_j - 2\mathbf{x}_i^T\mathbf{x}_j^T.$$

It is easy to check that:

$$(\mathbf{X}\mathbf{X}^T)_{ij} = \mathbf{x}_i\mathbf{x}_j^T.$$

Consider $\hat{\mathbf{x}} = \frac{1}{2}[\mathbf{x}_1^T\mathbf{x}_1, \dots, \mathbf{x}_n^T\mathbf{x}_n]^T$. We have:

$$\mathbf{1}_n\hat{\mathbf{x}}^T = \begin{bmatrix} \frac{1}{2}\mathbf{x}_1^T\mathbf{x}_1 & \dots & \frac{1}{2}\mathbf{x}_n^T\mathbf{x}_n \\ \frac{1}{2}\mathbf{x}_1^T\mathbf{x}_1 & \dots & \frac{1}{2}\mathbf{x}_n^T\mathbf{x}_n \\ \vdots & \ddots & \vdots \\ \frac{1}{2}\mathbf{x}_1^T\mathbf{x}_1 & \dots & \frac{1}{2}\mathbf{x}_n^T\mathbf{x}_n \end{bmatrix}$$

This means that $(\mathbf{1}_n\hat{\mathbf{x}}^T)_{ij} = \frac{1}{2}\mathbf{x}_j^T\mathbf{x}_j$ and moreover $(\hat{\mathbf{x}}\mathbf{1}_n^T)_{ij} = \frac{1}{2}\mathbf{x}_i^T\mathbf{x}_i$

Therefore:

$$\left(-\frac{1}{2}\mathbf{D}^{(2)}(\mathbf{X})\right)_{ij} = (\mathbf{X}\mathbf{X})_{ij} - (\mathbf{1}_n\hat{\mathbf{x}}^T)_{ij} - (\hat{\mathbf{x}}\mathbf{1}_n^T)_{ij}.$$

The element-wise identity implies the desired identity.

b) Since $-\frac{1}{2}\mathbf{E}_n\boldsymbol{\Delta}^{(2)}\mathbf{E}_n$ is non-negative definite and has the rank $\mathrm{rk}(-\frac{1}{2}\mathbf{E}_n\boldsymbol{\Delta}^{(2)}\mathbf{E}_n) \leq k$, it can be written as:

$$-\frac{1}{2}\mathbf{E}_n\boldsymbol{\Delta}^{(2)}\mathbf{E}_n = \sum_{i=1}^{k} \lambda_i \mathbf{v}_i \mathbf{v}_i^T,$$

where $\lambda_1 \geq \cdots \geq \lambda_k$ are top k eigenvalues of the matrix $-\frac{1}{2}\mathbf{E}_n\boldsymbol{\Delta}^{(2)}\mathbf{E}_n$ with corresponding orthonormal eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$. This can be obtained from spectral decomposition of $-\frac{1}{2}\mathbf{E}_n\boldsymbol{\Delta}^{(2)}\mathbf{E}_n$. Using this representation, the matrix $\mathbf{X}$ can be constructed as $\mathbf{X} = [\sqrt{\lambda_1}\mathbf{v}_1, \ldots, \sqrt{\lambda_k}\mathbf{v}_k]$. It can be seen that:

$$\mathbf{X}\mathbf{X}^T = \sum_{i=1}^{k} \lambda_i \mathbf{v}_i \mathbf{v}_i^T = -\frac{1}{2}\mathbf{E}_n\boldsymbol{\Delta}^{(2)}\mathbf{E}_n.$$

Moreover the image of $-\frac{1}{2}\mathbf{E}_n\boldsymbol{\Delta}^{(2)}\mathbf{E}_n$ is a subset of the image of $\mathbf{E}_n$. Therefore for all non-zero λ_i, the corresponding eigenvector $\mathbf{v}_i$ belongs to the image of $\mathbf{E}_n$ and since it is an orthogonal projection:

$$\mathbf{E}_n\mathbf{v}_i = \mathbf{v}_i.$$

If $\lambda_i = 0$, then trivially $\mathbf{E}_n\sqrt{\lambda_i}\mathbf{v}_i = \sqrt{\lambda_i}\mathbf{v}_i = 0$. This means that:

$$\mathbf{E}_n\mathbf{X} = \mathbf{X} \implies \mathbf{X}^T\mathbf{E}_n = \mathbf{X}^T.$$

c) The direction where $\mathbf{A} = 0$ is trivial. Let us assume $\mathbf{E}_n\mathbf{A}\mathbf{E}_n = 0$. This means that the matrix $\mathbf{A}$ takes each vector in the image of $\mathbf{E}_n$ to the kernel of $\mathbf{E}_n$. Note that the kernel of $\mathbf{E}_n$ is spanned by $\mathbf{1}_n$, so for each $\mathbf{v}$ such that $\mathbf{v}^T\mathbf{1}_n = 0$, we have:

$$\exists \alpha \in \mathbb{R}; \mathbf{A}\mathbf{v} = \alpha\mathbf{1}_n.$$

Pich $\mathbf{v} = \mathbf{e}_i - \mathbf{e}_j$. The equation above implies that $(\mathbf{A}\mathbf{v})_i = (\mathbf{A}\mathbf{v})_j$. But $(\mathbf{A}\mathbf{v})_k = a_{ki} - a_{kj}$. Therefore:

$$a_{ii} - a_{ij} = a_{ji} - a_{jj}.$$

But $a_{kk} = 0$ for all $1 \leq k \leq n$ and $\mathbf{A}$ is symmetric. Therefore $a_{ij} = 0$ for all i, j which means that $\mathbf{A} = 0$.

Solution of Problem 3

We start by expanding the following difference

$$(1+\beta)(1+\frac{\gamma}{\beta}) - (1+\sqrt{\gamma})^2 = 1 + \frac{\gamma}{\beta} + \beta + \gamma - (1 + 2\sqrt{\gamma} + \gamma) = \frac{\gamma}{\beta} + \beta - 2\sqrt{\gamma}$$
$$= \frac{\gamma - 2\beta\sqrt{\gamma} + \beta^2}{\beta} = \frac{(\sqrt{\gamma} - \beta)^2}{\beta}.$$

Since $\beta > 0$ we have that

$$(1+\beta)(1+\frac{\gamma}{\beta}) - (1+\sqrt{\gamma})^2 = \frac{(\sqrt{\gamma} - \beta)^2}{\beta} > 0,$$

yielding

$$(1+\beta)(1+\frac{\gamma}{\beta}) > (1+\sqrt{\gamma})^2$$

which proves the statement.