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Exercise 8
- Proposed Solution -

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Solution of Problem 1

Note that the discriminant rule is to allocate $\mathbf{x}$ to the group 1 if $|\mathbf{a}^T\mathbf{x}-\mathbf{a}^T\overline{\mathbf{x}}_\mathbf{1}| < |\mathbf{a}^T\mathbf{x}-\mathbf{a}^T\overline{\mathbf{x}}_\mathbf{2}|$ with $\mathbf{a}=\mathbf{W}^{-1}(\overline{\mathbf{x}}_\mathbf{1}-\overline{\mathbf{x}}_\mathbf{2})$. See that:

$$\mathbf{a}^T(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{1}) = \mathbf{a}^T(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{2}) + \mathbf{a}^T(\overline{\mathbf{x}}_\mathbf{2}-\overline{\mathbf{x}}_\mathbf{1}),$$

and note that since $\mathbf{W}^{-1}$ is nonnegative definite, we have:

$$\mathbf{a}^T(\overline{\mathbf{x}}_\mathbf{2}-\overline{\mathbf{x}}_\mathbf{1}) = (\overline{\mathbf{x}}_\mathbf{1}-\overline{\mathbf{x}}_\mathbf{2})^T\mathbf{W}^{-1}(\overline{\mathbf{x}}_\mathbf{2}-\overline{\mathbf{x}}_\mathbf{1}) \le 0,$$

hence $\mathbf{a}^T(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{1}) \le \mathbf{a}^T(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{2})$. We have three cases:

- If $\mathbf{a}^T(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{1}) > 0$, then $|\mathbf{a}^T\mathbf{x}-\mathbf{a}^T\overline{\mathbf{x}}_\mathbf{1}| < |\mathbf{a}^T\mathbf{x}-\mathbf{a}^T\overline{\mathbf{x}}_\mathbf{2}|$, and the discriminant rule implies that $\mathbf{x}$ is allocated to C_1.
- If $\mathbf{a}^T(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{2}) < 0$, then $|\mathbf{a}^T\mathbf{x}-\mathbf{a}^T\overline{\mathbf{x}}_\mathbf{1}| > |\mathbf{a}^T\mathbf{x}-\mathbf{a}^T\overline{\mathbf{x}}_\mathbf{2}|$, and the discriminant rule implies that $\mathbf{x}$ is allocated to C_2.
- If $\mathbf{a}^T(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{1}) > 0$ and $\mathbf{a}^T(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{1}) < 0$, the discriminant rule implies that $\mathbf{x}$ is allocated to C_1 if :
$$\mathbf{a}^T(-\mathbf{x}+\overline{\mathbf{x}}_\mathbf{1}) < \mathbf{a}^T(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{1}) \implies \mathbf{a}^T(2\mathbf{x}-\overline{\mathbf{x}}_\mathbf{2}-\overline{\mathbf{x}}_\mathbf{1}) > 0$$

Now just see that if $\mathbf{a}^T(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{1}) > 0$, then $\mathbf{a}^T(2\mathbf{x}-\overline{\mathbf{x}}_\mathbf{2}-\overline{\mathbf{x}}_\mathbf{1}) > 0$. And if $\mathbf{a}^T(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{2}) < 0$, then $\mathbf{a}^T(2\mathbf{x}-\overline{\mathbf{x}}_\mathbf{2}-\overline{\mathbf{x}}_\mathbf{1}) < 0$.

Another solution:
First of all, the discriminant rule can be simplified as follows:

$$\begin{aligned}
&|\mathbf{a}^T\mathbf{x}-\mathbf{a}^T\overline{\mathbf{x}}_\mathbf{1}| < |\mathbf{a}^T\mathbf{x}-\mathbf{a}^T\overline{\mathbf{x}}_\mathbf{2}| \implies\\
&(\mathbf{a}^T\mathbf{x}-\mathbf{a}^T\overline{\mathbf{x}}_\mathbf{1})^2 < (\mathbf{a}^T\mathbf{x}-\mathbf{a}^T\overline{\mathbf{x}}_\mathbf{2})^2 \implies\\
&(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{1})^T\mathbf{a}\mathbf{a}^T(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{1}) < (\mathbf{x}-\overline{\mathbf{x}}_\mathbf{2})^T\mathbf{a}\mathbf{a}^T(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{2}).
\end{aligned}$$

Note that:

$$\begin{aligned}
(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{1})^T\mathbf{a}\mathbf{a}^T(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{1}) &= (\mathbf{x}-\overline{\mathbf{x}}_\mathbf{2}+\overline{\mathbf{x}}_\mathbf{2}-\overline{\mathbf{x}}_\mathbf{1})^T\mathbf{a}\mathbf{a}^T(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{2}+\overline{\mathbf{x}}_\mathbf{2}-\overline{\mathbf{x}}_\mathbf{1})\\
&= (\mathbf{x}-\overline{\mathbf{x}}_\mathbf{2})^T\mathbf{a}\mathbf{a}^T(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{2}) + (\overline{\mathbf{x}}_\mathbf{2}-\overline{\mathbf{x}}_\mathbf{1})^T\mathbf{a}\mathbf{a}^T(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{1})\\
&+ (\mathbf{x}-\overline{\mathbf{x}}_\mathbf{2})^T\mathbf{a}\mathbf{a}^T(\overline{\mathbf{x}}_\mathbf{2}-\overline{\mathbf{x}}_\mathbf{1})\\
&= (\mathbf{x}-\overline{\mathbf{x}}_\mathbf{2})^T\mathbf{a}\mathbf{a}^T(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{2}) + (\overline{\mathbf{x}}_\mathbf{2}-\overline{\mathbf{x}}_\mathbf{1})^T\mathbf{a}\mathbf{a}^T(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{1})\\
&+ (\overline{\mathbf{x}}_\mathbf{2}-\overline{\mathbf{x}}_\mathbf{1})^T\mathbf{a}\mathbf{a}^T(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{2})\\
&= (\mathbf{x}-\overline{\mathbf{x}}_\mathbf{2})^T\mathbf{a}\mathbf{a}^T(\mathbf{x}-\overline{\mathbf{x}}_\mathbf{2}) - (\overline{\mathbf{x}}_\mathbf{1}-\overline{\mathbf{x}}_\mathbf{2})^T\mathbf{a}\mathbf{a}^T(2\mathbf{x}-\overline{\mathbf{x}}_\mathbf{1}-\overline{\mathbf{x}}_\mathbf{2})
\end{aligned}$$

Using this equlity in the discriminant rule, we obtain the rule as:

$$(\overline{\mathbf{x}}_\mathbf{1} - \overline{\mathbf{x}}_\mathbf{2})^T \mathbf{a}\mathbf{a}^T(2\mathbf{x} - \overline{\mathbf{x}}_\mathbf{1} - \overline{\mathbf{x}}_\mathbf{2}) > 0.$$

However since $\mathbf{W}^{-1}$ is nonnegative definite (see above), $(\overline{\mathbf{x}}_\mathbf{1} - \overline{\mathbf{x}}_\mathbf{2})^T\mathbf{a} > 0$ and therefore it suffices that:

$$\mathbf{a}^T(2\mathbf{x} - \overline{\mathbf{x}}_\mathbf{1} - \overline{\mathbf{x}}_\mathbf{2}) > 0.$$

Solution of Problem 2

The ML discriminant rule for classification into two classes C_1 and C_2 allocates $\mathbf{x}$ to C_1 if:

$$f_1(\mathbf{x}) > f_2(\mathbf{x}),$$

or equivalently if:

$$(\mathbf{x} - \boldsymbol{\mu}_1)^T\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}_1) < (\mathbf{x} - \boldsymbol{\mu}_2)^T\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}_2).$$

Note that:

$$\begin{aligned}
(\mathbf{x} - \boldsymbol{\mu}_1)^T\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}_1) &= (\mathbf{x} - \boldsymbol{\mu}_2 + \boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)^T\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}_2 + \boldsymbol{\mu}_2 - \boldsymbol{\mu}_1) \\
&= (\mathbf{x} - \boldsymbol{\mu}_2)^T\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}_2) + (\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)^T\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}_1) \\
&+ (\mathbf{x} - \boldsymbol{\mu}_2)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1) \\
&= (\mathbf{x} - \boldsymbol{\mu}_2)^T\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}_2) + (\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)^T\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}_1) \\
&+ (\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)^T\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}_2) \\
&= (\mathbf{x} - \boldsymbol{\mu}_2)^T\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}_2) - (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)^T\boldsymbol{\Sigma}^{-1}(2\mathbf{x} - \boldsymbol{\mu}_1 - -\boldsymbol{\mu}_2)
\end{aligned}$$

Using this equlity in the discriminant rule, we have:

$$(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)^T\boldsymbol{\Sigma}^{-1}(2\mathbf{x} - \boldsymbol{\mu}_1 - -\boldsymbol{\mu}_2) > 0,$$

which is the desired expression.

Solution of Problem 3

Note that $\mathbf{B} = \sum_{l=1}^{g} n_l(\overline{\mathbf{x}}_\mathbf{l} - \overline{\mathbf{x}})(\overline{\mathbf{x}}_\mathbf{l} - \overline{\mathbf{x}})^T$ and $\mathbf{W} = \sum_{l=1}^{g} \mathbf{X}_l^T\mathbf{E}_l\mathbf{X}_l$. But the crucial identity for this problem is the followin:

$$\mathbf{S} = \mathbf{B} + \mathbf{W}.$$

First of all, let $(\lambda, \mathbf{v})$ be eigenvalue-eigenvector pair for the matrix $\mathbf{W}^{-1}\mathbf{B}$. We have:

$$\mathbf{W}^{-1}\mathbf{S} = \mathbf{W}^{-1}\mathbf{B} + \mathbf{I} \implies \mathbf{W}^{-1}\mathbf{S}\mathbf{v} = \mathbf{W}^{-1}\mathbf{B}\mathbf{v} + \mathbf{v} = (\lambda + 1)\mathbf{v}.$$

Therefore $(\lambda + 1, \mathbf{v})$ is an eigenvalue-eigenvector pair for $\mathbf{W}^{-1}\mathbf{S}$. Moreover it can be seen that

$$\mathbf{W}^{-1}\mathbf{S}\mathbf{v} = (\lambda + 1)\mathbf{v} \implies v = (\lambda + 1)\mathbf{S}^{-1}\mathbf{W}\mathbf{v},$$

which means that $(\frac{1}{\lambda+1}, \mathbf{v})$ is an eigenvalue-eigenvector pair for $\mathbf{S}^{-1}\mathbf{W}$. Therefore the equivalence of three eigenvectors follow these discussions.