Problem 1. (Matrix Loewner Ordering Properties) Let $V$ and $W$ be two $n \times n$ non-negative definite matrices, such that $V = (v_{ij}) \preceq W = (w_{ij})$, with the eigenvalues as:

- $\lambda_1(V) \geq \cdots \geq \lambda_n(V)$,
- $\lambda_1(W) \geq \cdots \geq \lambda_n(W)$

Prove the following statements.

a) $\lambda_i(V) \leq \lambda_i(W)$, for $i = 1, \ldots, n$

b) $v_{ii} \leq w_{ii}$, for $i = 1, \ldots, n$

c) $v_{ii} + v_{jj} - 2v_{ij} \leq w_{ii} + w_{jj} - 2w_{ij}$

d) $\text{tr}(V) \leq \text{tr}(W)$

e) $\det(V) \leq \det(W)$

Problem 2. (Distribution of eigenvalues) Use Gerschgorin’s Theorem to find the smallest regions in which the eigenvalues of the matrix $A$ are concentrated. Is $A$ positive definite? Determine the smallest interval $[\lambda_{\min}, \lambda_{\max}]$ in which the real part of the eigenvalues are distributed.

$$A = \begin{pmatrix} 10 & 0.1 & 1 & 0.9 & 0 \\ 0.2 & 9 & 0.2 & 0.2 & \vdots \\ 0.3 & -0.1 & 5 + i & 0 & 0.1 \\ 0 & 0.6 & 0.1 & 6 & -0.3 \\ 0.3 & -0.3 & 0.1 & 0 & 1 \end{pmatrix}$$

Gerschgorin’s Theorem: Let $A \in \mathbb{C}^{n \times n}$, with entries $a_{ij}$, be given. For $i, j \in \{1, \ldots, n\}$ let $R_i = \sum_{j=1}^{n} |a_{ij}|$ and $C_j = \sum_{i=1}^{n} |a_{ij}|$ be the sum of the absolute values of the non-diagonal entries. Then every eigenvalue of $A$ lies within at least one of the discs centered at $a_{ii}$ with radius $\min\{R_i, C_i\}$.

Note that if one of the discs is disjoint from the others then it contains exactly one eigenvalue. If the union of $m$ discs is disjoint from the union of the other $n - m$ discs then the former union contains exactly $m$ and the latter $n - m$ eigenvalues of $A$. 
Problem 3. (Weights on A Leverage)
A beam has niches with distances $d_1 \geq \cdots \geq d_n$ from the pivot. There are $n$ weights of weight $w_1, \ldots, w_n$.

- How would you attach weights to niches so that torque of the beam is maximum?
- For any given assignment of weights to niches, can you improve the torque by exchanging the position of only two weights?