**Exercise 9**  
Friday, December 21, 2018

**Problem 1.** *(Fisher’s Linear Discriminant Function for two classes)* If Fisher’s linear discriminant function is used for classification into two classes $C_1$ and $C_2$, prove that an observation $x$ is allocated to $C_1$ if $a^T(x - \frac{1}{2}(\bar{x}_1 + \bar{x}_2)) > 0$ with $a = W^{-1}(\bar{x}_1 - \bar{x}_2)$.

**Problem 2.** *(ML Discriminant Rule for two classes)* Suppose that ML discriminant rule is used for classification into two classes $C_1$ and $C_2$. The class distributions are Gaussian and known as $N_p(\mu_1, \Sigma_1)$ and $N_p(\mu_2, \Sigma_2)$ with $\Sigma_1 = \Sigma_2 = \Sigma$. The densities are:

$$f_l(u) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(u - \mu_l)^T \Sigma^{-1}(u - \mu_l)\right\}, \quad u \in \mathbb{R}^p, l = 1, 2.$$ 

Prove that the ML rule allocates $x$ to the class $C_1$ if

$$\alpha^T(x - \mu) > 0,$$

where $\alpha = \Sigma^{-1}(\mu_1 - \mu_2)$ and $\mu = \frac{1}{2}(\mu_1 + \mu_2)$.

**Problem 3.** *(Eigenvalues in Fisher’s Linear Discriminant Analysis)* Let $X = [x_1, \ldots, x_n]^T \in \mathbb{R}^{n \times p}$ be samples and $W$ and $B$ are matrices corresponding to within-group and between-group sum of squares. Define $S = X^T E_n X$. Suppose that $W$ has rank $p$. Show that the following three eigenvectors are the same:

a) the eigenvector corresponding to the largest eigenvalue of $W^{-1}B$

b) the eigenvector corresponding to the largest eigenvalue of $W^{-1}S$

c) the eigenvector corresponding to the smallest eigenvalue of $S^{-1}W$