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Tutorial 1
- Proposed Solution -

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Solution of Problem 1

Entropy of a random experiment/event can be written as $H(\mathbf{p}) = -\sum_i p_i \log p_i$, where p_i is the probability of the i-th outcome.

a)

$$\begin{aligned} H(\mathbf{p}) &= -0.9 * \log_2(0.9) - 0.05 * \log_2(0.05) - 0.05 * \log_2(0.05) \\ &= 0.568 \text{ bits.} \end{aligned}$$

b)

$$\begin{aligned} H(\mathbf{p}) &= -0.4 * \log_2(0.4) - 0.3 * \log_2(0.3) - 0.3 * \log_2(0.3) \\ &= 1.570 \text{ bits.} \end{aligned}$$

Here, you can observe the second experiment is more uncertain (random) compared to the first experiment.

Solution of Problem 2

We wish to find all probability vectors $\mathbf{p} = (p_1, p_2, ...p_n)$ which minimize

$$H(\mathbf{p}) = -\sum_i p_i \log p_i$$

Now $-p_i \log p_i \geq 0$,with equality if and only if $p_i = 0$ or 1. Hence the only possible probability vectors which minimize $H(\mathbf{p})$ are those with $p_i = 1$ for some i and $p_j = 0$, $j \neq i$. There are n such vectors, i.e ., $(1, 0, .., 0)..., (0, ..., 0, 1)$, and the minimum value of $H(\mathbf{p})$ is 0.

Solution of Problem 3

From basic probability theory, we know

$$P(X = x_i) = \sum_j P(X = x_i, Y = y_j).$$

and

$$P(X = x_i | Y = y_j) = \frac{P(X = x_i, Y = y_j)}{P(Y = y_j)} = \frac{P(X = x_i, Y = y_j)}{\sum_i P(X = x_i, Y = y_j)}.$$

a)

$$\begin{aligned} H(X) &= -\sum_i P(X = x_i) \log P(X = x_i) \\ &= -\sum_i \Big(\sum_j P(X = x_i, Y = y_j) \log \sum_j P(X = x_i, Y = y_j) \Big) \\ &= -\frac{2}{3} \log \frac{2}{3} - \frac{1}{3} \log \frac{1}{3} = 0.918 \text{ bits.} \end{aligned}$$

Similarly,

$$\begin{aligned} H(Y) &= -\sum_j P(Y = y_j) \log P(Y = y_j) \\ &= -\frac{1}{3} \log \frac{1}{3} - \frac{2}{3} \log \frac{2}{3} = 0.918 \text{ bits.} \end{aligned}$$

b)

$$\begin{aligned} H(X|Y) &= -\sum_{i,j} P(X = x_i, Y = y_j) \log P(X = x_i | Y = y_j) \\ &= -\frac{1}{3} \log 1 - 0 \log 0 - \frac{1}{3} \log \frac{1}{2} - \frac{1}{3} \log \frac{1}{2} \\ &= 0.667 \text{ bits.} \end{aligned}$$

Similarly,

$$H(Y|X) = -\sum_{i,j} P(X = x_i, Y = y_j) \log P(Y = y_j | X = x_i) = 0.667 \text{ bits.}$$

c)

$$H(X, Y) = -\sum_{i,j} P(x_i, y_j) \log P(x_i, y_j) = 3 \times -\frac{1}{3} \log \frac{1}{3} = 1.585 \text{ bits}$$

d)

$$H(Y) - H(Y|X) = 0.251 \text{ bits.}$$

Solution of Problem 4

Entropy of function of random variable.

a) $H(X, g(X)) = H(X) + H(g(X)|X)$ by the chain rule for entropies.

b) $H(g(X)|X) = 0$ since for any particular value of X, $g(X)$ is fixed, and hence $H(g(X)|X) = \sum_x p(x) H(g(X)|X = x) = \sum_x 0 = 0$.

c) $H(X, g(X)) = H(g(X)) + H(X|g(X))$ again by the chain rule.

d) $H(X|g(X)) \geq 0$, with equality if and only if X is a function of $g(X)$, i.e., $g(.)$ is one-to-one. Hence $H(X, g(X)) \geq H(g(X))$

Combining parts (b) and (d), we obtain $H(X) \geq H(g(X))$.

Solution of Problem 5

Let $y = g(x)$. Then

$$p(y) = \sum_{x:y=g(x)} p(x).$$

Consider any set of x's that map onto a single y. For this set

$$\sum_{x:y=g(x)} p(x) \log p(x) \leq \sum_{x:y=g(x)} p(x) \log p(y) = p(y) \log p(y),$$

since log is a monotone increasing function and $p(x) \leq \sum_{x:y=g(x)} p(x) = p(y)$, extending this argument to the entire range of X (and Y), we obtain

$$\begin{aligned} H(X) &= -\sum_x p(x) \log p(x) \\ &= -\sum_y \sum_{x:y=g(x)} p(x) \log p(x) \\ &\geq -\sum_y p(y) \log p(y) \\ &= H(Y), \end{aligned}$$

with equality if and only if g is one-to-one with probability one.

a) $Y = 2^X$ is one-to-one and hence the entropy, which is just a function of the probabilities (and not the values of a random variable) does not change, i.e., $H(X) = H(Y)$.

b) $Y = cos(X)$ is not necessarily one to one. Hence all that we can say is that $H(X) \geq H(Y)$, with equality if cosine is one-to-one on the range of X.

Solution of Problem 6

The number X of tosses till the first head appears has the geometric distribution with parameters $p = 1/2$, where $P(X = n) = pq^{n-1}$, $n \in \{1, 2, ...\}$, $q = 1 - p$. Hence the entropy of X is

$$\begin{aligned}
H(X) &= -\sum_{n=1}^{\infty} pq^{n-1}\log(pq^{n-1}) \\
&= -\sum_{n=1}^{\infty} pq^{(n-1)}\log p - \sum_{n=1}^{\infty}(n-1)pq^{(n-1)}\log q \\
&= -\left[\sum_{n=0}^{\infty} pq^{n}\log p + \sum_{n=0}^{\infty} npq^{n}\log q\right] \\
&= \frac{-p\log p}{1-q} - \frac{pq\log q}{p^2} \\
&= \frac{-p\log p - q\log q}{p} \\
&= H(p)/p \text{ bits}
\end{aligned}$$

if $p = 1/2$, then $H(X) = 2$ bits.