Recapulation:

\[ X^n = (X_1, \ldots, X_n) \in \mathbb{R}^n, \quad X_1, \ldots, X_n \text{ i.i.d. } \sim p(x) \]

\[ X^n \xrightarrow{\text{encoder}} f_n(X^n) \in \{1, 2, \ldots, 2^n R\} \xrightarrow{\text{decoder}} \hat{X}^n \]

\[ \hat{X}^n = (\hat{X}_1, \ldots, \hat{X}_n) \in \mathbb{R}^n \]

**Def. 5.3.** Distortion measure

\[ d(X^n, \hat{X}^n) = \frac{1}{n} \sum_{i=1}^{n} d(x_i, \hat{x}_i) \]

**Def. 5.4.** A \( (2^n R, n) \)-rate distortion code of rate \( R \) and block length \( n \) consists of an encoder

\[ f_n : \mathbb{X}^n \rightarrow \{1, 2, \ldots, 2^n R\} \]

and a decoder

\[ g_n : \{1, 2, \ldots, 2^n R\} \rightarrow \mathbb{X}^n. \]

The expected distortion of the \((f_n, g_n)\) is

\[ D = E d(X^n, \hat{X}^n) = E d(X^n, g_n(f_n(X^n))). \]
Remarks:

a) $X, X'$ are assumed to be finite.
b) $2^{nR}$ means $\lceil 2^{nR} \rceil$, if it is not integer.
c) $f_n$ yields $2^{nR}$ different values.

We need $\approx nR$ bits to represent each. Hence,

$$R = \text{no. of bits per source symbol needed to represent } R_n(X^n).$$

d) $D = E[d(X^n, X')]$

$$= E[d(X^n, g_n(f_n(X^n)))]
= \sum_{X^n \in X^n} p(x^n) d(X^n, g_n(f_n(x^n)))$$

e) $\{g_n(1), \ldots, g_n(2^{nR})\}$ is called codebook.

$f_n^{-1}(1), \ldots, f_n^{-1}(2^{nR})$ are called assignment regions.

Ultimate goal of lossy source coding:
- minimize $R$ for a given $D$
- minimize $D$ for a given $R$. 
Def. 5.5. A rate distortion pair \((R,D)\) is called achievable if there exists a sequence of \((2^{nR},n)\)-rate distortion codes such that

\[
\lim_{n \to \infty} Ed(X^n, gu(R_n(X^n))) \leq D.
\]

Def. 5.6. The rate distortion function is defined as

\[
R(D) = \inf \{ R \mid (R,D) \text{ is achievable} \}.
\]

Def. 5.7. The information rate rate distortion function \(R^*_E(D)\) is defined as follows

\[
R^*_E(D) = \min_{p(\hat{x}|x)} \sum_{(x,\hat{x})} p(x,\hat{x}) d(x,\hat{x}) \leq D
\]

\[
= \min_{p(\hat{x}|x)} \sum_{(x,\hat{x})} p(x,\hat{x}) d(x,\hat{x}) \leq D
\]

Compare with capacity, 

\(C\) : given \(p(x|\hat{x})\), \(\max I(X,\hat{X})\) over the input alphabet

\(R^*_E(D)\) : given \(p(x)\), \(\min I(X,\hat{X})\) over "channels" s.t.

the expected distortion does exceed \(D\).
Theorem 5.8.

a) $R_I(D)$ is a convex nonincreasing fkt. of $D$.

b) $R_I(D) = 0$, if $D > D^*$, $D^* = \min_{R \in \mathbb{R}} E_d(X, \hat{x})$

c) $R_I(0) \leq H(X)$


Theorem 5.9. $X \sim \text{Bin}(1, p)$, $P(X=0) = 1-p$, $P(X=1) = p$, $0 \leq p \leq 1$, $d : \text{Hamming distance}$.

$$R_E(D) = \begin{cases} H(p) - H(D), & 0 \leq D \leq \min\{p, 1-p\} \\ 0, & \text{otherwise} \end{cases}$$

Proof. W.l.o.g. assume $p < \frac{1}{2}$, otherwise interchange 0 and 1.

$$R_E(D) = \min_{p(X|X)} I(X; \hat{X})$$

Assume $D \leq p < \frac{1}{2}$

$$I(X; \hat{X}) = H(X) - H(X|\hat{X})$$

$\hat{X}$ given $X$ has the same entropy as $X$ given $\hat{X}$

$$\geq H(X) - H(X \oplus \hat{X})$$

$\geq H(p) - H(D)$

$$\frac{P(X \oplus \hat{X} = 1)}{P(X \oplus \hat{X})} = P(X = \hat{X})$$

$\leq D < \frac{1}{2}$
This lower bound is obtained by the following joint distribution of $(X, \hat{X})$:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$\hat{X}$</th>
<th>$\hat{X}$ = 0</th>
<th>$\hat{X}$ = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \frac{D(p-D)}{1-2D} )</td>
<td>( \frac{1-D(p-D)}{1-2D} )</td>
<td>1-p</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{D(1-p-D)}{1-2D} )</td>
<td>( \frac{1-D(1-p-D)}{1-2D} )</td>
<td>p</td>
</tr>
<tr>
<td></td>
<td>( \frac{1-p-D}{1-2D} )</td>
<td>( \frac{p-D}{1-2D} )</td>
<td>1</td>
</tr>
</tbody>
</table>

Corresponds to a BSC:

\[
\begin{array}{c|c|c|c}
1-p-D & 1-p & 1-p-D & 1-D(p-D) \\
1-2D & \hline & 1-D & \hline & 1-2D \\
\end{array}
\]

It follows that:

\[
P(X \neq \hat{X}) = Ed(X, \hat{X})
= \frac{D(p-D)}{1-2D} + \frac{D(1-p-D)}{1-2D} = D
\]

Further:

\[
I(X; \hat{X}) = H(X) - H(X|\hat{X})
= H(p) - \left[ H(X|\hat{X} = 0) P(\hat{X} = 0) + H(X|\hat{X} = 1) P(\hat{X} = 1) \right]
= H(p) - H(D)
= H(p) - H(D)
\]

Such that the lower bound is obtained.
If \( D \geq p \) set \( P(X^2 = 0) = 1 \) and set

\[
\begin{array}{c|c|c|c|c}
X & 0 & 1 & 1-p & 2-p \\
- & 0 & 1 & 0 & 0 \\
1 & p & 0 & 0 & p \\
1 & 0 & 0 & 1 & 0
\end{array}
\]

Then  
\[ E(X, X^2) = P(X \neq X^2) = P(X = 1) = p \leq D \]

and

\[
I(X; X^2) = H(X) - H(X | X^2) = H(p) - H(X | X^2 = 0) = H(p) - H(0) = 0
\]

Plot for \( \text{Bin}(1, \frac{1}{2}) \):
Theorem 5.10. (Course to the rate distortion theorem)

\[ R(\delta) \geq R_{\text{E}}(\delta) \]

Proof:
Recall the general situation: \( X_1, \ldots, X_n \) iid, \( nX \sim p(x), x \in \mathcal{X} \).
\( \hat{X}^n = g_n(\hat{f}_n(X^n)) \) has at most \( 2^{nR} \) values. Hence

\[ H(\hat{X}^n) \leq \log 2^{nR} = nR \]

We first show: \((R, D)\) achievable \(\Rightarrow R \geq R_{\text{E}}(\delta)\)
Suppose \((R, D)\) is achievable. Then

\[ nR \geq H(\hat{X}^n) \]

\[ \geq H(\hat{X}^n) - H(\hat{X}^n | X^n) \]

\[ = I(\hat{X}^n, X^n) - I(X^n, \hat{X}^n) \]

\[ = H(X^n) - H(X^n | \hat{X}^n) \]

\[ = \sum_{i=1}^{n} H(X_i) - \sum_{i=1}^{n} H(X_i | \hat{X}_i, (X_{i-1}, X_{i+1})) \]

\[ \geq \sum_{i=1}^{n} I(X_i, \hat{X}_i) \]

\[ \geq \sum_{i=1}^{n} R_{\text{E}}(\text{Ed}(X_i, \hat{X}_i)) \]

\[ = n \sum_{i=1}^{n} \frac{1}{n} R_{\text{E}}(\text{Ed}(X_i, \hat{X}_i)) \geq n R_{\text{E}}(\frac{1}{n} \sum_{i=1}^{n} \text{Ed}(X_i, \hat{X}_i)) \]

\[ = n R_{\text{E}} \text{Ed}(X^n, \hat{X}^n) \geq n R_{\text{E}}(D) \]

\[ \text{inf} \ R = R(D) \geq R_{\text{E}}(D) \]

Hence, \((R, D)\) achievable.
The reverse inequality in Th. 5.10 also holds.

**Th. 5.11. (The Rate Distortion Theorem)**

\[ R(D) = R_I(D). \]

**Proof.** \( R(D) \geq R_I(D) \quad \text{Th. 5.10.} \)

\[ R_I(D) \leq R(D) \]

Yeung: Section 9.5, p. 206-212

Cover & Thomas: Section 10.5, p. 318-324