Solution of Problem 1

We have a generator \( a \equiv g^{\frac{p-1}{q}} \mod p \), with \( g \in \mathbb{Z}_p^* \), \( q \mid p - 1 \), \( p, q \) prime and \( a \neq 1 \). By definition of the order of a group, we know that:

\[
a^{\text{ord}_p(a)} \equiv 1 \mod p.
\]

Recall: \( \text{ord}_p(a) = \min\{k \in \{1, \ldots, \varphi(p)\} \mid a^k \equiv 1 \mod p\} \). With \( a \neq 1 \rightarrow \text{ord}_p(a) > 1 \). Next, we compute \( a^q \) and substitute \( g^{\frac{p-1}{q}} \):

\[
a^q \equiv \left(g^{\frac{p-1}{q}}ight)^q \equiv g^{p-1} \equiv 1 \mod p.
\]

From this we obtain \( 1 < \text{ord}_p(a) \leq q \).

Yet to show: Does a \( k \in \mathbb{Z} \) with \( k < q \) exist so that \( k \) is the order of the group?

This is a proof by contradiction.

Assume the subgroup has indeed \( k = \text{ord}_p(a) < q \), i.e., \( \exists k < q : k = \text{ord}_p(a) \). Then:

\[
a^q \equiv a^{lk+r}, \; l \in \mathbb{Z}, \; r < k, \\
\equiv a^r \\
\equiv 1 \mod p.
\]

We distinguish two possible cases:

- \( \text{ord}_p(a) \nmid q \Rightarrow a^r \equiv 1 \mod p, \) with \( 1 < r < \text{ord}_p(a) \) (Def. of \( \text{ord}_p(a) \))
- \( \text{ord}_p(a) \mid q \Rightarrow a^0 \equiv 1 \mod p \checkmark \)

Since \( q \) is prime \( \Rightarrow \text{ord}_p(a) \mid q \) there are only two divisors of \( q \), namely 1 and \( q \):

- \( \text{ord}_p(a) = 1 \checkmark \) (since \( a \neq 1 \) is assumed)
- \( \text{or} \text{ord}_p(a) = q \checkmark \) (We obtain \( k = q \) and not the demanded \( k < q \))

The cyclic subgroup has order \( q \) in \( \mathbb{Z}_p^* \), if \( a \) is chosen according to the algorithm.
Solution of Problem 2

Choose a pair \((\tilde{u}, \tilde{v}) \in \mathbb{Z} \times \mathbb{Z}\) such that \(\gcd(\tilde{v}, q) = 1\), so that \(\tilde{v}\) is invertible modulo \(q\).

The forged signature is constructed by:

\[
\begin{align*}
  r &\equiv (a^\tilde{u} \mod p) \mod q, \\
  s &\equiv r\tilde{v}^{-1} \mod q,
\end{align*}
\]

Then \((r, s)\) is a valid signature for the message \(m = s\tilde{u} \mod q\).

Check verification procedure of the DSA:

1. Check \(0 < r < q, 0 < s < q\). ✓ (due to modulo \(q\))
2. Compute \(w \equiv s^{-1} \mod q\).
3. In this step, no hash-function is used by the given assumption, i.e., \(h(m) = m\):
   \[
   \begin{align*}
   u_1 &\equiv wm \equiv s^{-1}s\tilde{u} \equiv \tilde{u} \mod q, \\
   u_2 &\equiv rw \equiv rs^{-1} \mod q.
   \end{align*}
   \]
4. \(v = a^{u_1} a^{u_2} \equiv a^{u_1+uxs^{-1}} \equiv a^{u_1+ux} \equiv a^{\tilde{u}(a^x)^{\tilde{v}}} \equiv (a^\tilde{u} \mod p) \mod q\).
5. The forged DSA signature is valid, since \(v = r\) holds.

Solution of Problem 3

a) We demand the following conditions on the two prime parameters \(p\) and \(q\):

   i) \(2^{159} < q < 2^{160}\),
   ii) \(2^{1023} < p < 2^{1024}\),
   iii) \(q \mid p - 1\).

We use a stepwise approach going through i), ii), and iii).

Our suggested algorithm to find a pair of primes \(p\) and \(q\) is:

1) Get a random odd number \(q\) with \(2^{159} < q < 2^{160}\).
2) Repeat step 1) if \(q\) is not prime. (e.g., use the Miller-Rabin Primality Test)
3) Get a random even number \(k\) with \(\left\lfloor \frac{2^{1023}-1}{q} \right\rfloor < k < \left\lceil \frac{2^{1024}-1}{q} \right\rceil\) and set \(p = kq + 1\).
4) If \(p\) is not prime, repeat step 3).

Check if the algorithm finds a correct pair of primes \(p, q\) according to i), ii), and iii):

- With step 1), \(2^{159} < q < 2^{160}\) holds, as demanded in i). ✓
- Due to step 2), \(q\) is prime. ✓
- Due to step 3), it holds:
  \[
  p = kq + 1 \gtrless \left\lfloor \frac{2^{1023}-1}{q} \right\rfloor q + 1 \gtrless 2^{1023},
  \]
  \[
  p = kq + 1 \gtrless \left\lceil \frac{2^{1024}-1}{q} \right\rceil q + 1 \leq 2^{1024},
  \]
  and therefore \(2^{1023} < p < 2^{1024}\) holds, as demanded in ii). ✓
• Step 3) also provides \( p = kq + 1 \Leftrightarrow q \mid p - 1 \), as demanded in iii).

An even \( k \) ensures that \( p \) is an odd number.

• Step 4) provides that \( p \) is also prime.

Altogether, the proposed algorithm works.

b) In steps 2) and 4), a primality test is chosen (here: Miller-Rabin Primality Test), such that the error probability for a composite \( q \) is negligible.

The success probability of finding a prime of size \( x \) is about \( \frac{1}{\ln(x)} \). (cf. hint)

If even numbers (these are obviously not prime) are skipped, the success probability doubles. The success probability of finding a single prime is estimated by:

\[
p_{\text{succ, } p} \approx 2 \cdot \frac{|\{p \in \mathbb{Z} \mid p \leq n, \ p \text{ prime}\}|}{n}.
\]

The combined probability of success for a pair of primes \( p \) and \( q \) is approximately:

\[
= \frac{2}{\ln(2^{100})} \cdot \frac{2}{\ln(2^{1024})} = \frac{1}{88,512 \ln(2)^2} \approx 5.08 \cdot 10^{-5}.
\]