Problem 1. (Dividers) Let $a, b, c, d \in \mathbb{Z}$. The integer $a$ divides $b$, if and only if there exists a $k \in \mathbb{Z}$ such that $a \cdot k = b$. This property is denoted by $a \mid b$. Prove the following implications.

a) $a \mid b$ and $b \mid c \implies a \mid c$.

b) $a \mid b$ and $c \mid d \implies (ac) \mid (bd)$.

c) $a \mid b$ and $a \mid c \implies a \mid (xb + yc) \ \forall x, y \in \mathbb{Z}$.

Problem 2. (GCD Multiplicativity) Let $a, b, m \in \mathbb{Z}$ and $\gcd(a, b)$ the greatest common divisor of $a$ and $b$.

a) Show the following.

$$\gcd(a, b) = 1 \implies \gcd(ab, m) = \gcd(a, m) \gcd(b, m)$$

b) Show that the reverse direction does not hold true.

Problem 3. (Scytale) For the encryption with an ancient Scytale, a parchment is wrapped around a wand such that there are $l \in \mathbb{N}$ rows and $k \in \mathbb{N}$ columns, cf. the conceptual figure. The letters of the plaintext $m = (m_1, m_2, \ldots, m_{kl})$ are written columnwise on the parchment. After unwrapping, the cryptogram is given on the stripe of parchment.

a) Give the entries $\pi(i)$ for $i \in \{1, 2, l, l + 1, (k - 1)l + 1, kl - 1, kl\}$ for the permutation

$$\pi = \begin{pmatrix}
1 & 2 & \ldots & l & l + 1 & \ldots & (k - 1)l + 1 & \ldots & kl - 1 & kl \\
\pi(1) & \pi(2) & \ldots & \pi(l) & \pi(l + 1) & \ldots & \pi((k - 1)l + 1) & \ldots & \pi(kl - 1) & \pi(kl)
\end{pmatrix},$$

which describes the encryption scheme of the Scytale with $l$ rows and $k$ columns.