# Homework 2 in Optimization in Engineering

Prof. Dr. Rudolf Mathar, Simon Görtzen, Markus Rothe

25.04.2012

**Exercise 1.** (properties of convex sets) A set $\mathcal{C}$ is convex, if

$$\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y} \in \mathcal{C} \text{ for all } \boldsymbol{x}, \boldsymbol{y} \in \mathcal{C},\ \alpha \in [0,1].$$

**a)** Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be convex sets. Show that $\mathcal{C}_1 \cap \mathcal{C}_2$ is convex.

**b)** Prove the following equivalence: A set $\mathcal{C} \subseteq \mathbb{R}^n$ is convex if and only if the intersection of $\mathcal{C}$ and any line in $\mathbb{R}^n$ is convex.

**Exercise 2.** (convex hull) The *convex hull* $\operatorname{conv}(\mathcal{S})$ of a set $\mathcal{S}$ is the set of all convex combinations of (a finite number of) points in $\mathcal{S}$:

$$\operatorname{conv}(\mathcal{S}) = \left\{ \sum_{i=1}^{k} \alpha_i \boldsymbol{x}_i \,\middle|\, \sum_{i=1}^{k} \alpha_i = 1,\ \boldsymbol{x}_i \in \mathcal{S},\ \alpha_i \geq 0,\ 1 \leq i \leq k,\ k \in \mathbb{N} \right\}$$

Show that $\operatorname{conv}(\mathcal{S})$ is the intersection of all convex sets which include $\mathcal{S}$:

$$\operatorname{conv}(\mathcal{S}) = \bigcap_{\substack{\mathcal{C} \text{ convex} \\ \text{with } \mathcal{S} \subseteq \mathcal{C}}} \mathcal{C}$$

**Exercise 3.** (convex figures) Show that the following sets are convex.

**a)** A slab $\left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \alpha \leq \boldsymbol{a}^T \boldsymbol{x} \leq \beta \right\}$ with $\boldsymbol{a} \in \mathbb{R}^n_{\neq \boldsymbol{0}}$ und $\alpha,\ \beta \in \mathbb{R}$.

**b)** A (hyper)rectangle $\{\boldsymbol{x} \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i,\ 1 \leq i \leq n\}$ with $\alpha_i,\ \beta_i \in \mathbb{R},\ 1 \leq i \leq n$.

**c)** A wedge $\left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \boldsymbol{a}_1^T \boldsymbol{x} \leq \beta_1,\ \boldsymbol{a}_2^T \boldsymbol{x} \leq \beta_2 \right\}$ with $\boldsymbol{a}_1,\ \boldsymbol{a}_2 \in \mathbb{R}^n_{\neq \boldsymbol{0}}$ and $\beta_1,\ \beta_2 \in \mathbb{R}$.

**Reminder:** *Halfspaces* $\left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \boldsymbol{a}^T \boldsymbol{x} \leq \beta \right\}$ with $\boldsymbol{a} \in \mathbb{R}^n_{\neq \boldsymbol{0}}$ and $\beta \in \mathbb{R}$ are convex.