Exercise 1. (partial sum of convex sets) Show that if $S_1$ and $S_2$ are convex sets in $\mathbb{R}^{m+n}$, then so is their partial sum

$$S = \{(x, y_1 + y_2) \mid x \in \mathbb{R}^m, y_1, y_2 \in \mathbb{R}^n, (x, y_1) \in S_1, (x, y_2) \in S_2\}.$$ 

Exercise 2. (invertible linear-fractional functions) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the linear-fractional function

$$f(x) = \frac{Ax + b}{c^T x + d}, \quad \text{dom } f = \{x \mid c^T x + d > 0\}.$$ 

Suppose the matrix

$$Q = \begin{bmatrix} A & b \\ c^T & d \end{bmatrix}$$

is nonsingular. Show that $f$ is invertible and that $f^{-1}$ is again a linear-fractional function. Give an explicit expression for $f^{-1}$ in terms of $Q$.

Exercise 3. (convex norm cone) Let $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. Suppose $||\cdot||$ is any norm in $\mathbb{R}^n$. The norm cone associated with this norm is the set

$$S = \{(x, t) \mid ||x|| \leq t\}.$$ 

Show that the norm cone is convex.