Exercise 1. (convex and concave functions) Decide which of the following functions are convex or concave and give reasons.

a) \( f(x) = e^x - 1, \quad x \in \mathbb{R} \)

b) \( f(x) = x_1 x_2, \quad x \in \mathbb{R}^2_0 \)

c) \( f(x) = \frac{1}{x_1 x_2}, \quad x \in \mathbb{R}^2_0 \)

d) \( f(x) = e^{x_1^2 + x_2^2}, \quad x \in \mathbb{R}^2 \)

Exercise 2. (epigraph) Let \( f: \mathcal{C} \to \mathbb{R} \) be a function defined on a convex, non-empty set \( \mathcal{C} \subseteq \mathbb{R}^n \). Show that \( f \) is convex if and only if the epigraph of \( f \)

\[
epi(f) = \{(x, y) \in \mathcal{C} \times \mathbb{R} | f(x) \leq y\}
\]

is a convex set.

Exercise 3. (separating convex and concave functions) Suppose \( f: \mathbb{R}^n \to \mathbb{R} \) is a convex function and \( g: \mathbb{R}^n \to \mathbb{R} \) is a concave function such that \( g(x) \leq f(x) \) for all \( x \). Show that there exists an affine function \( h: \mathbb{R}^n \to \mathbb{R} \) with \( g(x) \leq h(x) \leq f(x) \) for all \( x \). In other words, if a concave function \( g \) is an underestimator of a convex function \( f \), then we can fit an affine function between \( f \) and \( g \).