Exercise 1. (exact line search) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the quadratic function

$$f(x) = \frac{1}{2} (x_1^2 + \gamma x_2^2)$$

with $\gamma > 0$. Show that the minimization of $f$ using a descent method with exact line search and start point $x^{(0)} = (\gamma, 1)$ leads in the $k^{th}$ iteration to

$$x_1^{(k)} = \gamma \left( \frac{\gamma - 1}{\gamma + 1} \right)^k,$$

$$x_2^{(k)} = \left( -\frac{\gamma - 1}{\gamma + 1} \right)^k.$$

Exercise 2. (convergence behaviour of Newton method) Newton’s method with fixed step size $t = 1$ can diverge if the initial point is not close to $x^*$. In this problem we consider two examples.

a) $f(x) = \log (e^x + e^{-x})$ has a unique minimizer $x^* = 0$. Run Newton’s method with fixed step size $t = 1$, starting at $x^{(0)} = 1$ and at $x^{(0)} = 1.1$.

b) $f(x) = -\log x + x$ has a unique minimizer $x^* = 1$. Run Newton’s method with fixed step size $t = 1$, starting at $x^{(0)} = 3$.

Plot $f$ and $f'$, and show the first few iterates.