Exercise 1. (Separation Theorem) Complete the proof of the Separation Theorem (Theorem 2.1 in the lecture notes), which is proved for a special case in the lecture. Show that a separating hyperplane exists for two disjoint convex sets $C$ and $D$.

Hints.

- You can use the result proved in Theorem 2.1 in the lecture, i.e., that a separating hyperplane exists when there exist points in the two sets whose distance is equal to the distance between the two sets.
- If $C$ and $D$ are disjoint convex sets, then the set $\{x - y \mid x \in C, y \in D\}$ is convex and does not contain the origin.

Exercise 2. (Supporting hyperplanes) Represent each of the following closed, convex sets $C \subseteq \mathbb{R}^2$ as an intersection of halfspaces.

(a) $C = \{x \in \mathbb{R}^2 \mid x_2 \geq e^{x_1}\}$.

(b) $C = \{x \in \mathbb{R}^2_{>0} \mid x_1x_2 \geq 1\}$.

Exercise 3. (Linear-fractional functions and convex sets) Let $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ be the linear fractional function

$$f(x) = \frac{Ax + b}{c^T x + d}, \quad \text{dom} f = \{x \in \mathbb{R}^n \mid c^T x + d > 0\}.$$  

The inverse image of a convex set $C$ under $f$ is defined as

$$f^{-1}(C) = \{x \in \text{dom} f \mid f(x) \in C\}.$$  

Give a description of the inverse image $f^{-1}(C)$ for each of the following sets $C \in \mathbb{R}^m$ as an intersection of the $\text{dom} f$ with a halfspace in (a), with a polyhedron in (b), and with an ellipsoid in (c).

(a) The halfspace $C = \{y \in \mathbb{R}^m \mid g^T y \leq h\}$ with $g \in \mathbb{R}^m_{\neq 0}$ and $h \in \mathbb{R}$.

(b) The polyhedron $C = \{y \in \mathbb{R}^m \mid G^T y \leq h\}$ with $G \in \mathbb{R}^m \times \mathbb{R}^n$ and $h \in \mathbb{R}^n$.

(c) The ellipsoid $C = \{y \in \mathbb{R}^m \mid y^T P^{-1} y \leq 1\}$ where $P \in \mathbb{S}^n_{>0}$. 