Exercise 1. (One-dimensional trust region problem) Consider the one-dimensional, real-valued trust region problem.

\[
\begin{align*}
\text{minimize} & \quad ax^2 + 2bx \\
\text{subject to} & \quad x^2 \leq 1.
\end{align*}
\]

(a) Determine all pairs \((a, b)\) for which the problem is non-convex.

In the following the problem shall be non-convex.

(b) Calculate the dual function \(L_D(\lambda)\)

(c) Give the optimal parameter \(\lambda^*\) which maximizes \(L_D\) and the corresponding value \(d^*\).

(d) Show that the optimal value of the primal problem \(p^*\) equals \(d^*\).
Exercise 2. (Geometric interpretation of duality) Consider the optimization problem

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_1(x) \leq 0
\end{align*}
\]

with \( f_0, f_1 : \mathbb{R} \to \mathbb{R} \), the sets

\[
\mathcal{G} = \{ (u, t) \mid \exists x \in D, f_0(x) = t, f_1(x) = u \} \quad \text{and} \\
\mathcal{A} = \{ (u, t) \mid \exists x \in D, f_0(x) \leq t, f_1(x) \leq u \}
\]

and the following realizations of the optimization problem.

(a) \( f_0(x) = x, f_1(x) = x^2 - 1 \)
(b) \( f_0(x) = x, f_1(x) = x^2 \)
(c) \( f_0(x) = x, f_1(x) = |x| \)
(d) \( f_0(x) = x, f_1(x) = \Gamma(x) \), with

\[
\Gamma(x) = \begin{cases} 
-x - 2, & x < -1 \\
x, & -1 \leq x \leq 1 \\
-x + 2, & x > 1
\end{cases}
\]
(e) \( f_0(x) = x^3, f_1(x) = -x + 1 \)
(f) \( f_0(x) = x^3, f_1(x) = -x + 1 \) and additional constraint \( x \geq 0 \)

What is the geometric interpretation of the given sets?

For all those realizations:

- formulate the dual problem,
- solve both the primal and the dual problem and
- answer the following three questions:
  - Is the problem convex?
  - Is Slater’s constraint qualification satisfied?
  - Does strong duality hold?