Exercise 1. (Optimality conditions) Consider the optimization problem

\[
\begin{align*}
    \text{minimize} & \quad x_1^2 + x_2^2 \\
    \text{subject to} & \quad (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1, \\
    & \quad (x_1 - 1)^2 + (x_2 + 1)^2 \leq 1
\end{align*}
\]

with variable \( x \in \mathbb{R}^2 \).

(a) Sketch the feasible set and level sets of the objective. Find the optimal point \( x^* \) and the optimal value \( p^* \).

(b) Give the expression of the associated Langrangian and state the KKT conditions. Do there exist Lagrange multipliers \( \lambda_1^* \) and \( \lambda_2^* \) that prove that \( x^* \) is optimal?

(c) Derive and solve the Lagrange dual problem. Does strong duality hold?

Exercise 2. (Gradient descent method with exact line search) The algorithms for unconstrained optimization problems in the lecture produce a minimizing sequence \( \{x^{(k)}\}_{k \in \mathbb{N}} \) where

\[
x^{(k+1)} = x^{(k)} + t^{(k)} \Delta x^{(k)}
\]

and \( t^{(k)} > 0 \). The vector \( \Delta x \) is called the step, the scalar \( t^{(k)} \) the step size, or step length. The methods discussed in the following are descent methods which means that

\[
f(x^{(k+1)}) < f(x^{(k)}),
\]

except when \( x^{(k)} \) is optimal.

A general descent method alternates between two steps, determining a descent direction \( \Delta x \), and the selection of a step size \( t \). The natural choice for the search direction is the negative gradient \( \Delta x = -\nabla f(x) \). The resulting algorithm is called gradient descent method. The step size in exact line search is determined by

\[
t = \arg \min_{s \geq 0} f(x + s \Delta x),
\]

in which \( t \) is chosen to minimize \( f \) along the ray \( \{x + s \Delta x \mid s \geq 0\} \).
Let $f: \mathbb{R}^2 \to \mathbb{R}$ be the quadratic function

$$f(x) = \frac{1}{2} (x_1^2 + \gamma x_2^2)$$

with $\gamma > 0$. Show that the minimization of $f$ using gradient descent method with exact line search and starting point $x^{(0)} = (\gamma, 1)$ leads in the $k$th iteration to

$$x_1^{(k)} = \gamma \left( \frac{\gamma - 1}{\gamma + 1} \right)^k,$$

$$x_2^{(k)} = \left( -\frac{\gamma - 1}{\gamma + 1} \right)^k.$$