On the Asymptotic Spectrum of the Error Probability of Composite Networks

Arash Behboodi
Dept. of Telecommunications, SUPELEC
91192 Gif-sur-Yvette, France
Email: {arash.behboodi}@supelec.fr

Pablo Piantanida
Dept. of Telecommunications, SUPELEC
91192 Gif-sur-Yvette, France
Email: {pablo.piantanida}@supelec.fr

***Abstract*—This paper investigates composite multiterminal networks, consisting of a set of multiterminal channels indexed or parametrized by a vector of channel parameters θ. The channel in operation is drawn from the sample set with probability $\mathbb{P}_\theta$. Instead of finding the maximum achievable rate subject to a –asymptotically– small error probability (EP), we look at the behavior of the error probability for a fixed coding rate. The asymptotic spectrum of error probability (ASEP) is then introduced as a novel and more general performance measure for composite networks. Indeed, the ASEP is defined as the smallest probability that the EP exceeds a desirable error ϵ for a coding rate r. It is shown that the ASEP is directly related to the ϵ-capacity of the network and assuming memoryless channels the ASEP can be bounded by a new region referred to as the *full error region*. Moreover, every code with a rate belonging to this region yields asymptotic EP equal to one.**

I. Introduction

Multiterminal networks such as computer networks, wireless sensor networks and Ad hoc networks are the essential part of modern telecommunication systems. Nevertheless the time-varying nature of wireless channels, e.g. due to fading and user mobility, does not allow the nodes to have full knowledge of all channel parameters involved in the communication. During years, an ensemble of works has been done on channel models for uncertainty. The compound channel, introduced by Wolfowitz [1], consists in a set of channels indexed by θ, $\mathcal{W}_\Theta = \{\mathbb{W}_\theta\}_{\theta\in\Theta}$ out of which the channel in operation is chosen and remains fixed during the communication (see [2], [3] and references therein). Unlike compound channels, a probability distribution is introduced over channels, i.e. θ, for averaged (or mixed) channels, discussed by Ahlswede [4] and further studied in [5], where the expected error probability is used as the reliability criteria. Ahlswede showed that the capacity of averaged semicontinuous stationary channels is in general larger than the corresponding compound channel and hence the statistical knowledge of channels at the transmitter appears in general to be beneficial. However, the weak capacity of averaged and compound channel can be non-positive for some channels, like slowly fading Gaussian channels where the worst channel has non-positive capacity. In order to deal with these scenarios, the notion of composite channels where the current channel is drawn according to a given probability distribution (PD) is used [6], [7]. The notion of *capacity versus outage* was introduced as the maximum achievable rate subject to a certain tolerable outage probability. It was shown [8] that this notion may not be precise enough to characterize the EP over channels that do not satisfy the strong converse property.

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In this paper, instead of finding the maximum achievable rate for an asymptotically small EP, we fix the desired coding rate r to study characteristics of the EP of the composite network. Indeed, the EP is taken as a random function of the channel parameters and then the notion of *asymptotic spectrum of EP* (ASEP) for a code C with rate r and EP $\epsilon \in [0, 1)$ is introduced. We show that most of measures of performance for composite networks can be derived from the ASEP. For fixed rate r and the error probability ϵ, the ASEP indicates the minimum possible probability that the asymptotic EP exceeds ϵ for a given r. The aim is to characterize the ASEP of composite memoryless networks. We derive the *full error region* for which every code with transmission rates in this region yields EP equal to one. More specifically, we prove that the closure of the cutset bounds on memoryless networks falls into the full error region. As a result of this, every memoryless network for which the cutset bound is achievable satisfies the strong converse property. Furthermore, if there is an unique code that achieves the capacity of all networks in the set, the asymptotic spectrum of EP coincides with the conventional notion of outage probability.

This paper is organized as follows. Main definitions are provided in Section II while Section III studies bounds on the ASEP. Finally, Section IV present an outline of the proofs.

Notation: The *information density* is defined by [3]

$$i(\mathbf{M}_n;\underline{\mathrm{Y}}) \triangleq \log \frac{\mathbb{P}_{Y^n|\mathbf{M}_n}(\underline{\mathrm{Y}}|\mathbf{M}_n)}{\mathbb{P}_{Y^n}(\underline{\mathrm{Y}})},$$

for an arbitrary sequence of n-dimensional outputs $\underline{\mathrm{Y}} = (Y_1, \dots, Y_n) \in \mathscr{Y}^n$ where $\mathbf{M}_n$ is an uniform RV over the index set $\mathcal{M}_n = \{1, \dots, M_n\}$. We will use $\limsup$ *in probability* of the random sequence Z_n defined by

$$\text{p-}\limsup_{n\to\infty} Z_n \triangleq \inf\left\{\beta : \lim_{n\to\infty} \Pr\{Z_n > \beta\} = 0\right\}.$$

II. Definitions and Problem Statement

The Composite Multiterminal Network (CMN) with m-nodes is characterized by a set $\mathcal{W}_\theta$ of conditional PDs

$$\left\{\mathbb{P}_{Y^n_{1\theta}\cdots Y^n_{m\theta}|X^n_{1\theta}\cdots X^n_{m\theta}} : \mathscr{X}^n_1 \times \ldots \times \mathscr{X}^n_m \longmapsto \mathscr{Y}^n_1 \times \ldots \times \mathscr{Y}^n_m\right\}$$

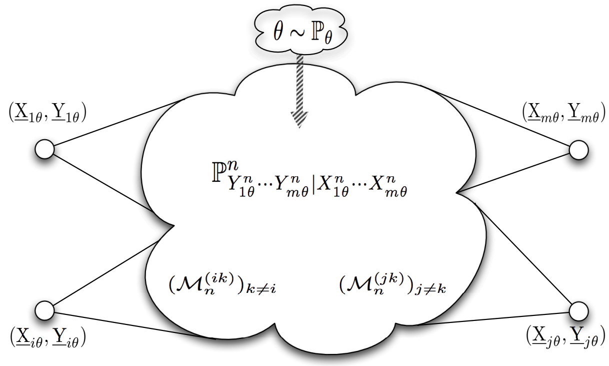

Fig. 1. Composite Multiterminal Network (CMN)

indexed with any vector of parameters $\theta \in \Theta$, and where each node $i = \{1, \ldots, m\}$ is equipped with a transmitter $\underline{X}_{i\theta} \in \mathscr{X}_i^n$ and a receiver $\underline{Y}_{i\theta} \in \mathscr{Y}_i^n$, as described in Fig. 1.

Each channel, denoted by $\mathbb{W}_\theta^n$, is assumed to be stationary and memoryless. Let $\mathbb{P}_\theta$ denote any arbitrary PD on the set of network parameters (or channel indices) Θ. Before the communication starts $\theta \in \Theta$ is assumed to be drawn from $\mathbb{P}_\theta$ remaining fixed during the entire transmission. The set $\mathcal{M}_n^{(ki)} \triangleq \{1, \ldots, M_n^{(ki)}\}$ represents the set of possible messages to be sent (in n channel uses) from source k to the i-th destination with $i \in \{1, \ldots, m\} \setminus \{k\}$. If there are no messages intended to node i from node k we set $\mathcal{M}_n^{(ki)} = \emptyset$.

Definition 1 (code and error probability): An $\big(n, M_n^{(kj)}, (\epsilon_{n,\theta})_{\theta\in\Theta}\big)$-code for the CMN consists of:

- A sequence of encoding mappings for $t = \{1, \ldots, n\}$ at each node $k \in \{1, \ldots, m\}$,
$$\varphi_{t,\theta}^{(k)} : \bigotimes_{i=\{1,\ldots,m\}\setminus\{k\}} \mathcal{M}_n^{(ki)} \otimes \mathscr{Y}_k^{t-1} \longmapsto \mathscr{X}_k$$
where $\mathcal{M}_n^{(ki)}$ is the message set from node k intended to destination node i, for every $i = \{1, \ldots, m\} \setminus \{k\}$.
- A decoder mapping at each node $k \in \{1, \ldots, m\}$,
$$\phi_{n,\theta}^{(jk)} : \mathscr{Y}_k^n \otimes \bigotimes_{i\in\{1,\ldots,m\}\setminus\{k\}} \mathcal{M}_n^{(ki)} \longmapsto \mathcal{M}_n^{(jk)}$$
for all source node $j \neq k \in \{1, \ldots, m\}$. Decoding sets corresponding to each decoding mapping are defined by $\mathcal{D}_{l,\theta}^{(jk)} \triangleq {\phi_{n,\theta}^{(jk)}}^{-1}(l)$ for all messages $l \in \mathcal{M}_n^{(jk)}$, which corresponds to the decoding sets for messages l intended to node k from node j.
- The error event $\mathcal{E}_\theta^{(jk)}(l) \triangleq \big\{Y_{k\theta}^n \notin \mathcal{D}_{l,\theta}^{(jk)}\big\}$ for all pairs $j \neq k \in \{1, ..., m\}$ and every $l \in \mathcal{M}_n^{(jk)}$ is defined as the event that the message l from node j cannot be correctly decoded at destination k. Therefore the error event for the network is the union of all error events $\mathcal{E}_\theta^{(jk)}(l)$ over all sources j and destinations k with messages l:
$$\epsilon_{n,\theta} \triangleq \Pr\left(\bigcup_{j\neq k} \bigcup_{l\in\mathcal{M}_n^{(jk)}} \left\{Y_{k\theta}^n \notin \mathcal{D}_{l,\theta}^{(jk)}, \mathbf{M}_n^{(jk)} = l\right\}\right).$$

For the rest we simply denote a code by C and emphasize that $\epsilon_{n,\theta}$ is a random variable. The EP from the node j to k, $\epsilon_{n,\theta}^{(jk)}$, can be similarly defined. It will always be presupposed that we are dealing with the kind of networks where full CSI is not available at each node.

A. Reliability Functions for Composite Networks

An alternative approach is the study of the behavior of error probability $\epsilon_{n,\theta}^{(jk)}$, $\epsilon_{n,\theta}$ as n goes to infinity for fixed rates. For the rest we assume that $\epsilon_{n,\theta}$ converges in distribution to ϵ_θ and since $\epsilon_{n,\theta}$ is uniformly integrable the limits remain intact.

Definition 2 (reliability functions): Given the tolerable error $0 \leq \epsilon < 1$ and a tuple of rates $\underline{r} = (r_{jk})_{j\neq k\in\{1,\ldots,m\}}$. Assume a $\big(n, M_n^{(jk)}, (\epsilon_{n,\theta})_{\theta\in\Theta}\big)$-code such that for all pairs $j \neq k \in \{1, \ldots, m\}$ we have
$$\liminf_{n\to\infty} \frac{1}{n} \log M_n^{(jk)} \geq r_{jk}.$$
The following reliability functions can be defined:

- The achievable EP for a code C is characterized by
$$\epsilon_{\text{-p}}(\underline{r}, \mathsf{C}) = \text{p-}\limsup_{n\to\infty} \epsilon_{n,\theta} \tag{1}$$
which means that the EP will be asymptotically less than the achievable EP $\epsilon_{\text{-p}}(\underline{r}, \mathsf{C})$ with probability 1. Notice that there may be no code satisfying $\lim_{n\to\infty} \mathbb{P}_\theta(\epsilon_{n,\theta} > \epsilon) = 0$ and thus for each $0 \leq \epsilon < 1$, there is non-zero probability that the error falls over it (e.g. Gaussian networks with fading). Condition in (1) can be relaxed to $\delta-$achievable
$$\epsilon_{\text{-}\delta}(\underline{r}, \mathsf{C}) = \delta\text{-}\limsup_{n\to\infty} \epsilon_{n,\theta} \tag{2}$$
where for any $0 \leq \delta < 1$
$$\delta\text{-}\limsup_{n\to\infty} \epsilon_{n,\theta} = \inf\left\{\alpha : \lim_{n\to\infty} \mathbb{P}_\theta(\epsilon_{n,\theta} > \alpha) \leq \delta\right\}.$$
- The average EP is characterized for a code C as follows
$$\bar{\epsilon}(\underline{r}, \mathsf{C}) = \lim_{n\to\infty} \mathbb{E}_\theta[\epsilon_{n,\theta}]. \tag{3}$$
This may lead to the achievability where ϵ is said to be achievable if there is a code C such that ϵ is larger than the average error, which implies the existence of codes with EP less than ϵ in $\mathcal{L}^1$ but not everywhere, meaning that for some $\theta \in \Theta$ the asymptotic EP may fall over $\bar{\epsilon}$. This shows that the average error used in [4] may not precise enough in general to characterize the EP, as it will be clear later.
- The throughput EP is defined for a code C by
$$\epsilon_T(\underline{r}, \mathsf{C}) = \sup_{0\leq\alpha<1} \lim_{n\to\infty} \alpha\, \mathbb{P}_\theta(\epsilon_{n,\theta} > \alpha). \tag{4}$$
This takes into account the probability that EP falls over the desired ϵ. Indeed, ϵ is said achievable if there is code C such that ϵ is larger than $\epsilon_T(\underline{r}, \mathsf{C})$.

It is particularly interesting to define *the smallest achievable EP* of a composite network by
$$\epsilon_{\text{-p}}(\underline{r}) = \inf_{\mathsf{C}} \epsilon_{\text{-p}}(\underline{r}, \mathsf{C}) = \inf_{\mathsf{C}} \text{p-}\limsup_{n\to\infty} \epsilon_{n,\theta}, \tag{5}$$

where the infimum is taken over all codes. This means that for ϵ smaller than $\epsilon_{\text{-p}}(\underline{r})$, there is a code such that we have:

$$\lim_{n\to\infty} \mathbb{P}_{\theta}(\epsilon_{n,\theta} > \epsilon) > 0.$$

Notice that the meaning of the smallest achievable EP is that if ϵ is bigger $\epsilon_{\text{-p}}(\underline{r})$, then information can be sent at rate $\underline{r}$ and EP less than ϵ with probability tending to 1. To avoid trivial results in some cases $\delta-$*smallest achievable EP* is defined by

$$\epsilon_{\text{-}\delta}(\underline{r}) = \inf_{\mathsf{C}} \epsilon_{\text{-}\delta}(\underline{r}, \mathsf{C}) = \inf_{\mathsf{C}} \delta\text{-} \limsup_{n\to\infty} \epsilon_{n,\theta} \tag{6}$$

where infimum is taken again over all codes. This means that for ϵ bigger than $\epsilon_{\text{-}\delta}(\underline{r})$, there is a code such that $\epsilon_{n,\theta}$ is less than ϵ with at least probability $1-\delta$.

Therefore, on one hand the expected EP (3) may not always be the adequate reliability function for CMNs, but on the other hand (1) may yield very pessimistic rates. The next section introduces a fundamental quantity, referred to as the *asymptotic spectrum of EP* (ASEP).

B. Asymptotic Spectrum of Error Probability (ASEP)

In the previous section, based on different criteria, we defined the smallest achievable EP for a fixed $\underline{r}$. We now investigate the asymptotic cumulative PD of EP for a fixed vector of transmission rates $\underline{r}$.

Definition 3 (asymptotic spectrum of EP): For every $0 \leq \epsilon \leq 1$ and transmission rates $\underline{r} = (r_{jk})_{j\neq k \in \{1,\dots,m\}}$, the asymptotic spectrum of EP for a code C, we denote

$$\mathcal{E}(\underline{r}, \epsilon, \mathsf{C}) = \lim_{n\to\infty} \mathbb{P}_{\theta}(\epsilon_{n,\theta} > \epsilon). \tag{7}$$

The asymptotic spectrum of EP for CMN is defined as:

$$\mathcal{E}(\underline{r}, \epsilon) = \inf_{\mathsf{C}} \lim_{n\to\infty} \mathbb{P}_{\theta}(\epsilon_{n,\theta} > \epsilon), \tag{8}$$

where the infimum is taken over all $\left(n, M_n^{(jk)}, (\epsilon_{n,\theta})_{\theta\in\Theta}\right)$-codes with rates satisfying

$$\liminf_{n\to\infty} \frac{1}{n} \log M_n^{(jk)} \geq r_{jk},$$

for all pairs $j \neq k \in \{1,\dots,m\}$.

Note that $\mathcal{E}(\underline{r}, \epsilon)$ indicates what is the smallest probability that the error falls over ϵ. The next proposition provides a relation between the ASEP and the previous notions introduced.

Proposition 1: For the CMN with transmission rates $\underline{r}$, the ASEP implies all reliability functions previously introduced.

- The smallest achievable (resp. to $\delta-$smallest) EP:

$$\epsilon_{\text{-p}}(\underline{r}) = \inf\left\{0 \leq \epsilon < 1 : \mathcal{E}(\underline{r}, \epsilon) = 0\right\},$$
$$\epsilon_{\text{-}\delta}(\underline{r}) = \inf\left\{0 \leq \epsilon < 1 : \mathcal{E}(\underline{r}, \epsilon) \leq \delta\right\}.$$

- The throughput EP of a code C

$$\epsilon_T(\underline{r}, \mathsf{C}) = \sup_{0\leq\epsilon<1} \epsilon\mathcal{E}(\underline{r}, \epsilon, \mathsf{C}).$$

- The expected EP of a code C

$$\bar{\epsilon}(\underline{r}, \mathsf{C}) = \int_0^1 \mathcal{E}(\underline{r}, \epsilon, \mathsf{C}) d\epsilon.$$

Proof: The proof of first three equalities follows directly from the definition. For the last inequality, using the fact that $\epsilon_{n,\theta}$ is positive and bounded we have:

$$\bar{\epsilon}(\underline{r}, \mathsf{C}) = \lim_{n\to\infty} \mathbb{E}[\epsilon_{n,\theta}] = \int_0^1 \lim_{n\to\infty} \mathbb{P}_{\theta}(\epsilon_{n,\theta} > t) dt$$

where uniform integrability and Lebesgue dominated convergence theorem is used to exchange limits. ■

III. Bounds on the Asymptotic Spectrum of EP

Consider first a non-composite network where transmission is at the rates $\underline{r}$. Then if a code achieves a EP ϵ its rate must necessarily belong to the ϵ-capacity region. Reciprocally, if the rate belongs to ϵ-capacity region, then there is a code such that it achieves EP ϵ. This leads to the next result.

Theorem 1: For the composite multiterminal network with the random parameter θ, it holds for every $0 \leq \epsilon < 0$ that

$$\mathbb{P}_{\theta}\Big(\limsup_{n\to\infty} \epsilon_{n,\theta} > \epsilon\Big) \geq \mathbb{P}_{\theta}(\underline{r} \notin \mathcal{C}_{\epsilon,\theta}), \tag{9}$$

where $\mathcal{C}_{\epsilon,\theta}$ is the ϵ-capacity of the network W_θ for a given θ.

Proof: According to the definition, for each θ, $\underline{r}$ is inside $\mathcal{C}_{\epsilon,\theta}$ if $\limsup_{n\to\infty} \epsilon_{n,\theta} \leq \epsilon$ which proves the theorem. ■

Notice that in the case of composite networks, a transmitter which is unaware of θ has a single code of fixed rate for all θ. Then, if for those θ for which the rate does not belong to $\mathcal{C}_{\epsilon,\theta}$ the EP $\epsilon_{n,\theta}$ will exceed ϵ. Whereas if the rate belongs to $\mathcal{C}_{\epsilon,\theta}$ then it is not guaranteed that $\epsilon_{n,\theta}$ will not exceed ϵ.

Suppose that transmitters fix their encoding function based on $\varphi_t^{(k)}$ and let ϕ be defined as the ensemble of such functions. For every θ and ϕ, let $\mathcal{R}_{\epsilon,\theta}(\phi)$ denote the ϵ-achievable region such as if the rate belongs to it, then the EP is less or equal than ϵ for the choice of ϕ. Clearly, we have the identity

$$\mathcal{E}(\underline{r}, \epsilon, \mathsf{C}) = \mathbb{P}_{\theta}(\underline{r} \notin \mathcal{R}_{\epsilon,\theta}(\phi)).$$

Corollary 1: For the error probability $\epsilon_{n,\theta}$ and ϵ-capacity defined as before, the asymptotic spectrum of EP satisfies:

$$\mathbb{P}_{\theta}(\underline{r} \notin \mathcal{C}_{\epsilon,\theta}) \leq \mathcal{E}(\underline{r}, \epsilon) \leq \inf_{\phi} \mathbb{P}_{\theta}(\underline{r} \notin \mathcal{R}_{\epsilon,\theta}(\phi)). \tag{10}$$

Remark 1: There exist composite networks, e.g. composite binary symmetric channels (CBSC), where a uniformly distributed code yields the best code for each channel in the set. In this case, we have the next identity:

$$\mathcal{E}(\underline{r}, \epsilon) = \mathbb{P}_{\theta}(\underline{r} \notin \mathcal{C}_{\epsilon,\theta}). \tag{11}$$

A. Composite Binary Symmetric Averaged Channel (CBSC)

A binary symmetric averaged channel (BSC) with three parameters is defined by three BSCs ($\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3$) with crossover probabilities $p_1 < p_2 < p_3 \leq \frac{1}{2}$ and weights $\alpha_1, \alpha_2, \alpha_3$ summing to one. The averaged channel is then defined as $\mathbb{B} = \alpha_1\mathbb{B}_1 + \alpha_2\mathbb{B}_2 + \alpha_3\mathbb{B}_3$. Denote by $\mathcal{C}(p) = 1 - H_2(p)$ the capacity of a BSC with parameter p. Kieffer [9] derived the capacity of this averaged BSC and showed that the channel does not satisfy the strong converse property. Moreover, the ϵ-capacity of this channel is characterized by

$$\mathcal{C}_\epsilon = \begin{cases} \mathcal{C}(p_3), & 0<\epsilon<\alpha_3 \\ \mathcal{C}(\lambda_{(p_2,p_3)}), & \epsilon=\alpha_3 \\ \mathcal{C}(p_2), & \alpha_3<\epsilon<\alpha_3+\alpha_2 \\ \mathcal{C}(\lambda_{(p_1,p_2)}), & \epsilon=\alpha_3+\alpha_2 \\ \mathcal{C}(p_1), & \alpha_3+\alpha_2<\epsilon<1 \end{cases} \tag{12}$$

where $\lambda_{(p_1,p_2)}$ is defined as $\lambda_{(p,q)} = \frac{\log\left(\frac{1-p}{1-q}\right)}{\log\left(\frac{1-p}{1-q}\right)+\log\left(\frac{q}{p}\right)}$.

Let us assume that there is additional randomness associated to this channel, and hence $\mathbf{p}_3$ takes random values between p_2 and $\frac{1}{2}$ with measure $\mathbb{P}_{\mathbf{p}_3}$ ($\theta = \mathbf{p}_3$). In addition to this assume that the source transmits a code with $r \leq \mathcal{C}(p_2)$. Therefore, the last three terms in the asymptotic spectrum of EP are automatically zero, which yields

$$\mathcal{E}(\underline{r},\epsilon) = \begin{cases} \mathbb{P}_{\mathbf{p}_3}\big(r>\mathcal{C}(\mathbf{p}_3)\big), & 0<\epsilon<\alpha_3 \\ \mathbb{P}_{\mathbf{p}_3}\big(r>\mathcal{C}(\lambda(p_2,\mathbf{p}_3))\big), & \epsilon=\alpha_3 \\ 0, & \alpha_3<\epsilon<1 \end{cases} \tag{13}$$

The smallest achievable EP writes as

$$\epsilon_{\text{p-}}(\underline{r}) = \inf\{0\leq\epsilon<1 : \mathcal{E}(\underline{r},\epsilon)=0\} \leq \alpha_3.$$

In other words, the EP is less than α_3 with probability 1 for $r<\mathcal{C}(p_2)$. On the other hand, the expected EP writes as

$$\bar{\epsilon}(\underline{r}) = \int_0^1 \mathcal{E}(\underline{r},\epsilon)d\epsilon = \alpha_3 \times \mathbb{P}_{\mathbf{p}_3}\big(r>\mathcal{C}(\mathbf{p}_3)\big).$$

Observe that the expected EP dismisses the information about the EP at the point $\epsilon=\alpha_3$. This implies that the expected EP is not enough general to provide a full characterization of the EP. Finally, the throughput EP writes as

$$\bar{\epsilon}_T(\underline{r}) = \sup_{0\leq\epsilon<1} \epsilon\mathcal{E}(\underline{r},\epsilon) = \alpha_3 \times \mathbb{P}_{\mathbf{p}_3}\big(r>\mathcal{C}(\lambda_{(p_2,\mathbf{p}_3)})\big).$$

Here the information about ϵ less than α_3 is lost in the above notion. This example clearly shows the relation between all these reliability notions and how the asymptotic spectrum of the EP can be used to derive them.

B. Outage Probability and Full Error Region

The main problem in characterizing the ASEP is that capacity is not known for most of multiterminal networks and consequently neither the ϵ-capacity. For instance, we need to focus on inner and outer bounds to delimitate the ASEP via other ways. The notion of outage probability P_{out} appears to be adequate. This is defined as the probability that a code with rate $\underline{r}$ cannot be correctly decoded $P_{\text{out}} = \mathbb{P}_\theta(\underline{r}\notin\mathcal{C}_\theta)$. Furthermore, if each channel θ satisfies the strong converse property ($\mathcal{C}_\theta = \mathcal{C}_{\epsilon,\theta}$ for $0\leq\epsilon<1$) and if there exists a unique best code for each channel in the set, it follows that the ASEP turns to be a Bernoulli trial with parameter P_{out}.

Denote by $\mathcal{R}_\theta(\phi)$ any achievable region known for each θ and ϕ. Then if the rate $\underline{r}$ is inside the region, the error probability tends to zero becoming less than ϵ, for any $0<\epsilon<1$. For fixed rate $\underline{r}$, measure of the channels with EP larger than ϵ is less or equal to the measure of channels with non-zero EP, implying that the ASEP is essentially less or equal than the probability that the rate $\underline{r}$. Similarly for the rate $\underline{r}$, number of those channels with the error probability bigger than ϵ is less or equal to probability of those channels with EP equal to one. Hence it is interesting to see for which values of $\underline{r}$, the EP tends to one. The following definition will be useful for the characterization of the asymptotic EP.

Definition 4 (full error region): Consider a multiterminal channel $\{\mathbb{W}^n\}_{n=1}^\infty$ with m sources and destinations. The full error region is the region $\mathcal{S}\subset\mathbb{R}_+^{m(m-1)}$ such that for all codes $\big(n, M_n^{(ij)}, \epsilon_n\big)$, if the rate vector

$$\underline{r} = \Big(\liminf_{n\to\infty}\frac{1}{n}\log M_n^{(ij)}\Big)$$

is inside the region $\mathcal{S}$ then $\lim_{n\to\infty}\epsilon_n = 1$.
The previous definition simply indicates that the EP becomes 1 for all nodes whose coding rate belong to this region.The following theorem provides converse limits of the ASEP.

Theorem 2: The CMN with random parameter θ satisfies

$$\mathbb{P}_\theta(\underline{r}\in\mathcal{S}_\theta) \leq \mathcal{E}(\underline{r},\epsilon) \leq \inf_\phi \mathbb{P}_\theta(\underline{r}\notin\mathcal{R}_\theta(\phi)), \tag{14}$$

where $\mathcal{R}_\theta$ is the achievable region of the network $\mathbb{W}_\theta$ and and $\mathcal{S}_\theta$ is the full error region of the corresponding channel θ.

The most general known outer bound for multiterminal networks is the cut-set bound [10], [11]. This states that any rate outside the region formed by cutset bound will have non-zero EP. In the next theorem, we prove that the error is necessarily one for any rate outside this region. This result provides an outer bound on the full error region.

Theorem 3: Consider a memoryless multiterminal network with m nodes. For any code $\big(n, M_n^{(ij)}, \epsilon_n\big)$ with rates $\underline{r} = \Big(\liminf_{n\to\infty}\frac{1}{n}\log M_n^{(ij)}\Big)$ that fall outside the region $\mathscr{S}_{\text{CB}}$ where:

$$\mathscr{S}_{\text{CB}} = co\bigcup_{P\in\mathcal{P}}\Big\{(R(\mathcal{S})\geq 0,\,\forall\mathcal{S}\subseteq\{1,2,\ldots,m\}) : \\ R(\mathcal{S}) < I(X_\mathcal{S};Y_{\mathcal{S}^c}|X_{\mathcal{S}^c})\Big\}$$

and $R(\mathcal{S}) = \sum_{i\in\mathcal{S},j\in\mathcal{S}^c} R_{ij}$, then $\lim_{n\to\infty}\epsilon_n = 1$.

Consider the composite finite field linear deterministic network where the channel in operation is chosen from a set of similar networks, indexed by $\theta\sim\mathbb{P}_\theta$. The nodes are divided into sources $\mathcal{T}$, destinations $\mathcal{D}$ and relays $\mathcal{R}$. Every channel has the cutset bound as capacity [12] and satisfies the strong converse property. Moreover, there is a unique optimum PD for all channels in the set. Then, the outage probability coincide with the asymptotic spectrum of EP of this network, as stated by the next corollary.

Corollary 2: For the composite finite field linear deterministic network, the ASEP for each $(\underline{r},\epsilon)$ is as follows:

$$\mathcal{E}(\underline{r},\epsilon) = \mathbb{P}_\theta(\underline{r}\notin\mathscr{C}_{\text{DN},\theta}), \tag{15}$$

where $\mathscr{C}_{\text{DN},\theta}$ is defined by

$$\Big\{(R(S)\geq 0) : R(S) < \min_{S\subseteq\mathcal{R}}\min_{d\in\mathcal{D}} H(Z_{S^c\theta}Y_{d\theta}|X_\mathcal{T}X_{S^c\theta})\Big\},$$

and the input distribution is chosen at each source to be independent and uniformly distributed.

It should be mentioned here that the r.h.s. of (15) is independent of ϵ, which means that the outage probability is a sufficient measure for the performance of this network.

IV. Outline of the Proof of Theorem 3

We present an outline of the proof of Theorem 3 but without proving the main lemmas. For sets $\mathcal{S}_1, \mathcal{S}_2 \subset \{1, \dots, m\}$ with cardinalities $\|\mathcal{S}_1\|$, $\|\mathcal{S}_2\|$, let $\mathbf{M}_n^{(\mathcal{S}_1\mathcal{S}_2)}$ be an $\|\mathcal{S}_1\|\|\mathcal{S}_2\|$-tuple $\left(\mathbf{M}_n^{(ij)}\right)_{i\in\mathcal{S}_1, j\in\mathcal{S}_2}$. The next result is Verdu-Han's lemma [3] extended to the multiterminal network.

Theorem 4: For all codes $\left(n, M_n^{(ij)}, \epsilon_n\right)$ with $i \neq j \in \{1, \dots, m\}$ the error probability satisfies

$$\epsilon_n \geq \Pr\left[\text{for some } \mathcal{S} \middle| \frac{1}{n} i(\mathbf{M}_n^{(\mathcal{S}\mathcal{S}^c)}; Y_{\mathcal{S}^c}^n) \leq \frac{1}{n} \log M_n^{(\mathcal{S}\mathcal{S}^c)} - \gamma \right] - (2^m - 1)\exp(-\gamma n).$$

For every $\gamma > 0$ and every $\mathcal{S} \subseteq \{1, \dots, m\}$, where $i(\mathbf{M}_n^{(\mathcal{S}\mathcal{S}^c)}; Y_{\mathcal{S}^c}^n)$ is the corresponding information density.

The general idea behind the proof is to show that for the memoryless network and the rate outside the closure of cut-set bound the probability on the right hand side tends to one for all $\gamma > 0$, as n goes to infinity. In other words, the information spectrum of $\frac{1}{n} i(\mathbf{M}_n^{(\mathcal{S}\mathcal{S}^c)}; Y_{\mathcal{S}^c}^n)$ is placed on the left-hand side of cut-set bound values. More precisely, the proof follows two general steps. First, it is shown that the information density is less –in measure– than another RV referred to $U_n^{(\mathcal{S})}$ as $n \to \infty$. In the next step, it is shown that $U_n^{(\mathcal{S})}$ is less –in measure– than the cut set bound. So if $\frac{1}{n}\log M_n^{(ij)} - \gamma$ is larger than the cutset bound as $n \to \infty$ then the probability in Theorem 4 will tend to 1 as $n \to \infty$.

A. First Step

For arbitrary inputs and outputs, define the quantity:

$$U_n^{(\mathcal{S})} = \frac{1}{n} \sum_{j=1}^{n} \log \frac{\mathbb{W}_j(Y_{\mathcal{S}^c j} | X_{\mathcal{S}^c j}, X_{\mathcal{S} j})}{\mathbb{P}_{Y_{\mathcal{S}^c j}, X_{\mathcal{S}^c j}}(Y_{\mathcal{S}^c j} | X_{\mathcal{S}^c j})},$$

where $\mathbb{W}_j$ is the j-th use of the channel. The channel is stationary and memoryless and then $\mathbb{W}_j$ remains constant, namely $\mathbb{W}$ for all j. Note that Y_k^n is not i.i.d. in general, but for simplicity we drop the subscripts when implicitly clear.

Lemma 1: For $U_n^{(\mathcal{S})}$ defined as above, we have

$$\lim_{n\to\infty} P\Big(\text{for all } S \big| U_n^{(\mathcal{S})} > \frac{1}{n} i(\mathbf{M}_n^{(\mathcal{S}\mathcal{S}^c)}; Y_{\mathcal{S}^c}^n)\Big) = 1. \tag{16}$$

B. Second Step

The second step is based on the following lemma.

Lemma 2: For $U_n^{(\mathcal{S})}$ defined as above, there is a probability distribution $\mathbb{P}_{X_\mathcal{S} X_{\mathcal{S}^c}}$ for each $x_\mathcal{N}^n$ such that:

$$\lim_{n\to\infty} \mathbb{E}\bigg[\Pr\Big(\text{for all } S \big| U_n^{(\mathcal{S})} < I(X_\mathcal{S}; Y_{\mathcal{S}^c} | X_{\mathcal{S}^c})\Big) \Big| X_\mathcal{N}^n\bigg] = 1,$$

Note that given $x_\mathcal{S}^n, x_{\mathcal{S}^c}^n$, the PD is $\mathbb{W}^n(Y_{\mathcal{S}^c}^n | x_\mathcal{S}^n, x_{\mathcal{S}^c}^n)$. Using the fact that the channel is memoryless, it can be seen that

$$\log \frac{\mathbb{W}_j(Y_{\mathcal{S}^c j} | x_{\mathcal{S}^c j}, x_{\mathcal{S} j})}{\mathbb{P}(Y_{\mathcal{S}^c j} | x_{\mathcal{S}^c j})}$$

are independent for all j, and we are faced with sum of independent RVs. Moreover, we have to assume that the channel $\mathbb{W}_j$ is the same for all j.

C. Final Step

To finalize the proof, the following inequality is needed:

$$\begin{aligned}
&\Pr\left[\text{for some } \mathcal{S} \middle| \frac{1}{n} i(\mathbf{M}_n^{(\mathcal{S}\mathcal{S}^c)}; Y_{\mathcal{S}^c}^n) \leq \sum_{i\in\mathcal{S}, j\in\mathcal{S}^c} \frac{1}{n}\log M_n^{(ij)} - \gamma \right] \\
&\geq \sum_{x_\mathcal{N}^n} \mathbb{P}_{X_\mathcal{N}^n}(x_\mathcal{N}^n) \Pr\Big(\text{for all } \mathcal{S} \big| U_n^{(\mathcal{S})} > \frac{1}{n} i(\mathbf{M}_n^{(\mathcal{S}\mathcal{S}^c)}; Y_{\mathcal{S}^c}^n), \\
&\qquad U_n^{(\mathcal{S})} < I(X_\mathcal{S}; Y_{\mathcal{S}^c} | X_{\mathcal{S}^c}) | x_\mathcal{N}^n\Big) \\
&\qquad \times \mathbf{1}\bigg[\left(\frac{1}{n}\log M_n^{(ij)}\right)_{i,j\in\mathcal{N}, i\neq j} \notin \mathscr{S}_{\text{CB}}\bigg].
\end{aligned}$$

Finally, from Theorem 4, the above lemmas and the last inequality by letting $n \to \infty$:

$$\lim_{n\to\infty} \epsilon_n \geq \lim_{n\to\infty} \mathbf{1}\bigg[\left(\frac{1}{n}\log M_n^{(ij)}\right)_{i,j\in\mathcal{N}, i\neq j} \notin \mathscr{S}_{\text{CB}}\bigg]. \tag{17}$$

Then, it is easy to check that the rates falling outside the closure of cutset bounds lead to error probability equal to 1.

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