Mixed Noisy Network Coding

Arash Behboodi
Dept. of Telecommunication Systems, TU Berlin
10587 Berlin, Germany
Email: behboodi@tkn.tu-berlin.de

Pablo Piantanida
Dept. of Telecommunications, SUPELEC
91192 Gif-sur-Yvette, France
Email: pablo.piantanida@supelec.fr

***Abstract*—Noisy Network Coding (NNC) was recently introduced, generalizing Compress-and-Forward (CF) to multiterminal networks. In this paper, we present Mixed Noisy Network Coding scheme as the generalization of NNC where part of the nodes are allowed to select Decode-and-Forward (DF) as their cooperative strategy while all nodes without exception transmit the compressed version of their observations. The compressed version of relays is exploited at each destination to decode the intended message. It is shown that Mixed NNC scheme performs potentially better than NNC. In particular, for AWGN networks it achieves a tighter "constant gap" with respect to the cut-set bound, provided that DF relays are chosen properly.**

I. Introduction

The relay channel is conceived as the building block of multiterminal cooperative networks. The main cooperative strategies for the single relay channel were first developed by Cover and El Gamal in 1979 [1], the attempts were made recently to develop similar cooperative strategies for multiterminal networks. Various configurations have been studied during years, including multiple access relay channel and broadcast relay channel [2]. Kramer *et al.* developed an inner bound for a point-to-point general network using Decode-and-Forward (DF) which achieves the capacity of the degraded multicast network [3]. Their scheme assumes certain hierarchy amount the relays that must be available at the source to generate the code. Whereas it is neither preferable nor likely that a certain hierarchy be always preserved during the communication given the heterogeneous nature of future wireless networks. The authors also present a generalized version of Compress-and-Forward (CF) for multiple relay networks.

On the other hand, El Gamal *et al.* developed an alternative version of CF in [4], later baptized as Noisy Network Coding. Although the scheme performs as well as conventional CF in single relay channel, it is strictly better when generalizing to multiple relay channel. Lim *et al.* in [5] proposed the Noisy Network Coding (NNC) scheme for the general multicast network which includes bounds in [6], [7]. These bounds are shown to be tight only for some specific cases like deterministic and erasure network but rarely for wireless models. Therefore, the authors in [5] showed that NNC inner bound achieves the cut-set bound within a constant gap that is independent of channel gains. NNC scheme utilizes "repetitive encoding", meaning that the same – long – message is transmitted in all communication blocks. The idea of sending different – short – messages in each block without performance loss was first used in [8] and formalized in [9] to then referred to Short Message Noisy Network Coding (SNNC). SNNC can be developed in different ways while in general the proof requires backward decoding. For instance, the transmission in [9] is done in $B+L$ blocks but to achieve the largest rate both B and L should tend to infinity. On the other hand in [10], L can be finite during which the last compression index is decoded. NNC and SNNC schemes have since then been exploited in various scenarios as in [9]–[11] amount many others.

SNNC scheme opens up the possibility of combining DF strategy with NNC. This path has been taken in previous contributions. The authors in [10] proposed Mixed Noisy Network Coding (MNNC) where relays are divided into two sets, relays in the first set use NNC while those in the second one use DF. Nevertheless, DF relays cannot help each other in decoding source messages and the destination can only benefit from the compression index of part of the relays since DF relays do not employ NNC strategy. This issue introduces considerable difficulty when attempting to compare this scheme with the original NNC. In this paper, we develop an achievable rate that generalizes NNC to the case of mixed coding strategy, where DF relays exploit the help of both NNC and DF relays. Mainly DF relays superimpose the compressed version of their observation on the source message. Transmission is done in $B+L$ blocks, where relays retransmit the compression index of block $B+2$ in the last $L-2$ blocks, and backward decoding is used at the destination. We derive a novel gap expression for MNNC for which it is shown that whenever DF is used, the gap respect to the cut-set bound cannot be made independent – constant – of the channel gains. However, provided that DF relays are chosen properly, the constant Gap is improved directly affected by the number of active DF relays.

This paper is organized as follows. Section II presents definitions while Section III introduces MNNC theorem, and the sketch of proof is relegated to Section IV. Finally, the constant gap for AWGN networks is presented in Section V.

II. Main Definitions

The unicast memoryless multiple relay network consists of a source X, a destination Y_1 and relays N denoted by pairs (X_i, Z_i) with $i \in \mathcal{N} = \{1, \dots, N\}$, and transition probability

$$\mathcal{W} = \Big\{ P_{Y_1 Z_1 \dots Z_N | X X_1 \dots X_N} : \mathcal{X} \times \mathcal{X}_1 \times \cdots \times \mathcal{X}_N \longmapsto \mathcal{Y}_1 \times \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_N \Big\}.$$

We shall denote $X_{\mathcal{S}} = \{X_i : i \in \mathcal{S}\}$ for any $\mathcal{S} \subseteq \mathcal{N}$.

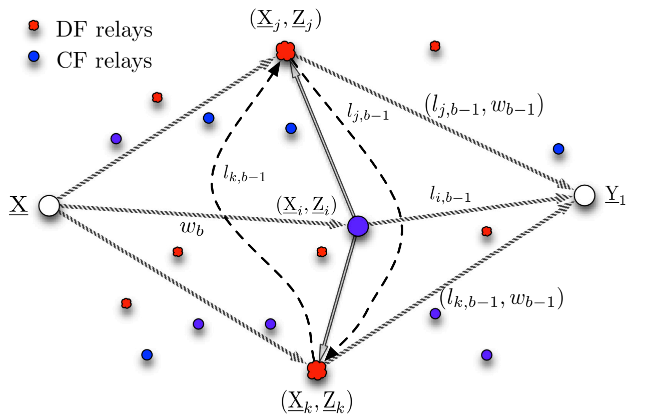

Fig. 1. Mixed Noisy Network Coding (MNNC).

Definition 1 (code and achievability): A code-$\mathcal{C}(n, M_n, \epsilon_n)$ for the unicast multiple relay channels consists of:

- An encoder mapping $\{\varphi : \mathcal{M}_n \longmapsto \mathcal{X}^n\}$,
- A decoder mapping $\{\phi : \mathcal{Y}_1^n \longmapsto \mathcal{M}_n\}$,
- A sequence of relay functions $\left\{f_i^{(k)} : \mathcal{Z}_k^{i-1} \longmapsto \mathcal{X}_k\right\}_{i=1}^n$

for $k \in \mathcal{N}$ and the average error probability is defined as

$$\epsilon_n \triangleq \Pr\left\{\phi(Y_1^n) \neq W\right\}.$$

A rate R is said to be achievable for the unicast multiple relay network if there exists a code-$\mathcal{C}(n, M_n, \epsilon_n)$ satisfying

$$\liminf_{n\to\infty} \frac{1}{n} \log M_n \geq R\,, \quad \text{and} \quad \limsup_{n\to\infty} \epsilon_n = 0.$$

The supremum of all achievable rates is the capacity of the unicast multiple relay network.

III. Mixed Noisy Network Coding

Consider the unicast multiple relay network described in Fig. 1. Nodes are divided in two groups, $\mathcal{V}$ and $\mathcal{V}^c$, where:

1) Relay nodes in set $\mathcal{V}^c$ employ partly DF and partly CF schemes as their cooperative strategy (e.g. nodes j, k) .
2) Relay nodes in set $\mathcal{V}$ use only CF (e.g. like i).

As it will be clarified later, the destination selects for decoding only the help of a subset of nodes $\mathcal{T} \subseteq \mathcal{N}$. Transmission is done in $B+L$ blocks where DF relays in each block $b \leq B+2$ forward the source message of the block $b-2$, as it is shown for nodes j, k in Fig. 1. Notice that this is slightly different from conventional DF scheme where the relay forwards the message of the previous block. The reason is that each relay in the network is transmitting the compressed version of its observation which introduces the possibility of full cooperation amount the network nodes. Particularly, DF relays can use the compression index of other relays to decode the source message. The compression index of the block b is transmitted only in block $b+1$, so DF relays have to wait until the end of block $b+1$ to decode the compression index and the message of the block b. Therefore, they can forward the message of block b only after block $b+1$. Similarly to the destination, the k-th DF relay in $\mathcal{V}^c$ exploits only the help of a selected subset $\mathcal{T}_k$ of relays. The following theorem provides the achievable rate corresponding to this communication strategy.

Theorem 1 (Mixed Noisy Network Coding): For the multiple relay network, the following rate is achievable:

$$R \leq \max_{P\in\mathcal{P}} \max_{\mathcal{V}\subseteq\mathcal{N}} \min\Bigg(\max_{\mathcal{T}\in\Upsilon(\mathcal{N})} \min_{\mathcal{T}-\mathcal{V}\subseteq\mathcal{S}\subseteq\mathcal{T}} R_{\mathcal{T}}(\mathcal{S}), \\ \min_{k\in\mathcal{V}^c} \max_{\mathcal{T}_k\in\Upsilon_k(\mathcal{N})} \min_{\mathcal{S}\subseteq\mathcal{T}_k} R_{\mathcal{T}_k}^{(k)}(\mathcal{S})\Bigg) \quad (1)$$

where

$$R_{\mathcal{T}}(\mathcal{S}) \triangleq I(XX_{\mathcal{S}};\hat{Z}_{\mathcal{S}^c}Y_1|X_{\mathcal{S}^c}Q) \\ -I(\hat{Z}_{\mathcal{S}};Z_{\mathcal{S}}|XX_{\mathcal{T}}\hat{Z}_{\mathcal{S}^c}Y_1Q) \quad (2)$$

$$R_{\mathcal{T}_k}^{(k)}(\mathcal{S}) \triangleq I(X;\hat{Z}_{\mathcal{T}_k}Z_k|VX_kX_{\mathcal{T}_k}Q) + I(X_{\mathcal{S}};Z_k|VX_kX_{\mathcal{S}^c}Q) \\ -I(\hat{Z}_{\mathcal{S}};Z_{\mathcal{S}}|VX_kX_{\mathcal{T}_k}\hat{Z}_{\mathcal{S}^c}Z_kQ). \quad (3)$$

where $\mathcal{S}^c \triangleq \mathcal{T} - \mathcal{S}$ in (2) and $\mathcal{S}^c \triangleq \mathcal{T}_k - \mathcal{S}$ in (3). Moreover, $\mathcal{T} \subseteq \mathcal{N}$, and $\mathcal{T}_k \subseteq \mathcal{N} - \{k\}$, and $\mathcal{V}^c = \mathcal{N} - \mathcal{V}$. Also $\Upsilon(\mathcal{N})$ and $\Upsilon_k(\mathcal{N})$ are defined as follows:

$$\Upsilon(\mathcal{N}) \triangleq \{\mathcal{T}\subseteq\mathcal{N} : \forall\mathcal{S}\subseteq\mathcal{T}, Q_{\mathcal{T}}(\mathcal{S}) \geq 0\}, \quad (4)$$

$$\Upsilon_k(\mathcal{N}) \triangleq \{\mathcal{T}\subseteq\mathcal{N}-\{k\} : \forall\mathcal{S}\subseteq\mathcal{T}, Q_{\mathcal{T}}^{(k)}(\mathcal{S}) \geq 0\} \quad (5)$$

with $Q_{\mathcal{T}}(\mathcal{S})$ and $Q_{\mathcal{T}}^{(k)}(\mathcal{S})$ defined by

$$Q_{\mathcal{T}}(\mathcal{S}) \triangleq I(X_{\mathcal{S}};\hat{Z}_{\mathcal{S}^c}Y_1|VXX_{\mathcal{S}^c}Q) \\ -I(\hat{Z}_{\mathcal{S}};Z_{\mathcal{S}}|VXX_{\mathcal{T}}\hat{Z}_{\mathcal{S}^c}Y_1Q),$$

$$Q_{\mathcal{T}}^{(k)}(\mathcal{S}) \triangleq I(X_{\mathcal{S}};Z_k|VX_kX_{\mathcal{S}^c}Q) \\ -I(\hat{Z}_{\mathcal{S}};Z_{\mathcal{S}}|VXX_kX_{\mathcal{T}}\hat{Z}_{\mathcal{S}^c}Z_kQ).$$

The set of all admissible distributions $\mathcal{P}$ is given by

$$\mathcal{P} = \Big\{P_{QVXX_{\mathcal{N}}Z_{\mathcal{N}}\hat{Z}_{\mathcal{N}}Y_1} = P_QP_{V|Q}P_{X|VQ}P_{Y_1Z_{\mathcal{N}}|XX_{\mathcal{N}}Q} \\ \prod_{j\in\mathcal{V}^c} P_{X_j|VQ}P_{\hat{Z}_j|VX_jZ_jQ} \prod_{j\in\mathcal{V}} P_{X_j|Q}P_{\hat{Z}_j|X_jZ_jQ}\Big\}.$$

The proof of this theorem is provided in next section. It can be shown using the same technique in [9] that the optimization of $R_{\mathcal{T}}(\mathcal{S})$ in (1) can be performed over $\mathcal{T} \subseteq \mathcal{N}$ instead of $\mathcal{T} \in \Upsilon(\mathcal{N})$. Theorem 1 reduces to SNNC by choosing $\mathcal{V} = \mathcal{N}$ as in [8], [9] which is also equivalent to NNC [5]. On the other hand, Theorem 1 differs from the cooperative mixed NNC in [10], where the relays who use DF are not allowed to use CF so cannot help each other when decoding messages. Therefore, Theorem 1 generalizes and includes all previous NNC schemes and it provides a potentially large achievable rate. It is worth mentioning that for the single degraded relay channel, it achieves the capacity which is not the case in NNC. Note that because DF relays has to use forward decoding, $R_{\mathcal{T}_k}^{(k)}(\mathcal{S})$ is smaller compared to the situation where backward decoding is employed. The reason for this, as stated in [8], is that the gain in NNC is achieved by delaying the decoding until the last block. But postponing decoding to the last block is not possible in DF relays which require decoding to re-encode in each block, introducing the rate loss we discussed.

IV. Outline of the Proof of Theorem 1

The relays in the network are divided into two sets $\mathcal{V}$ and $\mathcal{V}^c = \mathcal{N} - \mathcal{V}$. Those relays in $\mathcal{V}$ are using CF while the others are using both CF and DF. The DF relays transmit also the compressed version of their observation, superimposed over their DF code. The DF relay $k \in \mathcal{V}^c$ decodes the source message of block i using the compressed version of observation of other relays. Because the relays transmit the compression index of the block i in the block $i+1$, the k-th relay has to wait until the end of block $i+1$ to decode it and therefore DF relays has to wait until the block $i+2$ to forward the source message of the i-th block.

Moreover the relay k exploits only the compression index of relays in $\mathcal{T}_k \subseteq \mathcal{N} - \{k\}$. Similarly the destination decodes the compression index of the relays in $\mathcal{T} \subseteq \mathcal{N}$. The reason for this choice is that the performance may be degraded if the compression index of all relays is used. A set of relays contribute to augmenting the rate only if certain condition is satisfied. Each $\mathcal{T}_k$ and $\mathcal{T}$ consist of DF and CF relays. For simplicity, we adopt the following notation:

$$\mathcal{T}_k^{\text{DF}} \triangleq \mathcal{T}_k \cap \mathcal{V}^c\,,\;\; \mathcal{T}_k^{\text{CF}} \triangleq \mathcal{T}_k \cap \mathcal{V}\,,$$
$$\mathcal{T}^{\text{DF}} \triangleq \mathcal{T} \cap \mathcal{V}^c\,,\;\; \mathcal{T}^{\text{CF}} \triangleq \mathcal{T} \cap \mathcal{V}.$$

Code generation:

1) Randomly and independently generate 2^{nR} sequences $\underline{v}$ drawn i.i.d. from $P_V^n(\underline{v}) = \prod_{j=1}^n P_V(v_j)$. Index them as $\underline{v}(r)$ with index $r \in \left[1, 2^{nR}\right]$.
2) For each $k \in \mathcal{V}^c$ and each $\underline{v}(r)$, randomly and independently generate $2^{n\hat{R}_k}$ sequences $\underline{x}_k$ drawn i.i.d. from
$$P_{X_k|V}^n(\underline{x}_k|\underline{v}(r)) = \prod_{j=1}^n P_{X_k|V}(x_{k,j}|v_j(r)).$$
Index them as $\underline{x}_k(r, r_k)$, where $r_k \in \left[1, 2^{n\hat{R}_k}\right]$ for $\hat{R}_k = I(Z_k; \hat{Z}_k|X_k, V) + \epsilon$.
3) For each $k \in \mathcal{V}$, randomly and independently generate $2^{n\hat{R}_k}$ sequences $\underline{x}_k$ drawn i.i.d. from $P_{X_k}^n(\underline{x}_k) = \prod_{j=1}^n P_{X_k}(x_{k,j})$. Index them as $\underline{x}_k(r_k)$, where $r_k \in \left[1, 2^{n\hat{R}_k}\right]$ for $\hat{R}_k = I(Z_k; \hat{Z}_k|X_k) + \epsilon$.
4) For each $\underline{v}(r)$, randomly and conditionally independently generate 2^{nR} sequences $\underline{x}$ drawn i.i.d. from $P_{X|V}^n(\underline{x}|\underline{v}(r)) = \prod_{j=1}^n P_{X|V}(x_j|v_j)$. Index them as $\underline{x}(r, w)$, where $w \in \left[1, 2^{nR}\right]$.
5) For each $k \in \mathcal{V}^c$ and each $\underline{x}_k(r, r_k)$, randomly and conditionally independently generate $2^{n\hat{R}_k}$ sequences $\underline{\hat{z}}_k$ each with probability $P_{\hat{Z}_k|X_k}^n(\underline{\hat{z}}_k|\underline{x}_k(r, r_k)) = \prod_{j=1}^n P_{\hat{Z}_k|X_k}(\hat{z}_{k,j}|x_{k,j}(r, r_k))$. Index them as $\underline{\hat{z}}_k(r, r_k, \hat{s}_k)$, where $\hat{s}_k \in \left[1, 2^{n\hat{R}_k}\right]$.
6) For each $k \in \mathcal{V}$ and each $\underline{x}_k(r_k)$, randomly and conditionally independently generate $2^{n\hat{R}_k}$ sequences $\underline{\hat{z}}_k$ each with probability $P_{\hat{Z}_k|X_k}^n(\underline{\hat{z}}_k|\underline{x}_k(r_k)) = \prod_{j=1}^n P_{\hat{Z}_k|X_k}(\hat{z}_{k,j}|x_{k,j}(r_k))$. Index them as $\underline{\hat{z}}_k(r_k, \hat{s}_k)$, where $\hat{s}_k \in \left[1, 2^{n\hat{R}_k}\right]$.

Encoding part:

1) In every block $i = [1 : B]$, the source sends w_i using $\underline{x}\big(w_{(i-2)}, w_i\big)$ ($w_0 = w_{-1} = 1$). Moreover, for blocks $i = [B+1 : B+L]$, the source sends the dummy message $w_i = 1$ known to all users.
2) For every block $i = [1 : B+L]$, and each $k \in \mathcal{V}^c$, the relay k knows $w_{(i-2)}$ by assumption and $w_0 = w_{-1} = 1$, so it picks up $\underline{v}\big(w_{(i-2)}\big)$. For each $i = [1 : B+2]$, the relay k after receiving $\underline{z}_k(i)$, searches for at least one index $l_{k,i}$ with $l_{k,0} = 1$ such that
$$\big(\underline{v}(w_{(i-2)}), \underline{x}_k(w_{(i-2)}, l_{k,(i-1)}), \underline{z}_k(i), \underline{\hat{z}}_k(w_{(i-2)}, l_{k,(i-1)}, l_{k,i})\big) \in \mathcal{A}_\epsilon^n[V X_k Z_k \hat{Z}_k].$$
The probability of finding such $l_{k,i}$ goes to one as n goes to infinity due to the choice of $\hat{R}_k$.
3) For $i = [1 : B+2]$ and $k \in \mathcal{V}^c$, relay k knows from the previous block $l_{k,(i-1)}$ and $w_{(i-2)}$ and it sends $\underline{x}_k(w_{(i-2)}, l_{k,(i-1)})$. Moreover, relay k repeats $l_{k,(B+2)}$ for $i = [B+3 : B+L]$, i.e. for $L-2$ blocks.
4) For each $i = [1 : B+2]$, each $k \in \mathcal{V}$, the relay k after receiving $\underline{z}_k(i)$, searches for at least one index $l_{k,i}$ with $l_{k,0} = 1$ such that
$$\big(\underline{x}_k(l_{k,(i-1)}), \underline{z}_k(i), \underline{\hat{z}}_k(l_{k,(i-1)}, l_{k,i})\big) \in \mathcal{A}_\epsilon^n[X_k Z_k \hat{Z}_k].$$
The probability of finding such $l_{k,i}$ goes to one as n goes to infinity due to the choice of $\hat{R}_k$.
5) For $i = [1 : B+2]$ and $k \in \mathcal{V}$, relay k knows from the previous block $l_{k,(i-1)}$ and it sends $\underline{x}_k(l_{k,(i-1)})$. Moreover, relay k repeats $l_{k,(B+2)}$ for $i = [B+3 : B+L]$, i.e. for $L-2$ blocks.

Decoding part:

1) After transmission of block $i = [1 : B+1]$ and for each $k \in \mathcal{V}^c$, the relay k decodes the message w_i and the compression index $l_{\mathcal{T}_k,i}$, the compression index of relays in $\mathcal{T}_k$ for the block i, with the assumption that all messages and compression indices up to block $i-1$ have been correctly decoded. Note that there are two kind of relays inside $\mathcal{T}_k$, those who employ DF and those who are using CF. The relay k knows the message $w_{(i-2)}, w_{(i-1)}$ and so $\underline{v}\big(w_{(i-2)}\big)$ and $\underline{v}\big(w_{(i-1)}\big)$. Let's define the sequences:
$$\mathcal{E}_k(\hat{w}_b, \hat{l}_{\mathcal{T}_k,b}) = \Big\{\underline{x}(w_{(b-2)}, \hat{w}_b), \underline{v}(w_{(b-2)}), \underline{x}_k(w_{(b-2)}, l_{k,(b-1)}), \underline{z}_k(b), \Big(\underline{x}_k(l_{k,(b-1)}), \underline{\hat{z}}_k(l_{k,(b-1)}, \hat{l}_{k,b})\Big)_{k \in \mathcal{T}_k^{\text{CF}}}, \Big(\underline{x}_k(w_{(b-2)}, l_{k,(b-1)}), \underline{\hat{z}}_k(w_{(b-2)}, l_{k,(b-1)}, \hat{l}_{k,b})\Big)_{k \in \mathcal{T}_k^{\text{DF}}}\Big\}$$
$$\mathcal{E}_k(\hat{l}_{\mathcal{T}_k,b}) = \Big\{\underline{v}(w_{(b-1)}), \underline{x}_k(w_{(b-1)}, l_{k,b}), \underline{z}_k(b+1), \Big(\underline{x}_k(w_{(b-1)}, \hat{l}_{k,b})\Big)_{k \in \mathcal{T}_k^{\text{DF}}}, \Big(\underline{x}_k(\hat{l}_{k,b})\Big)_{k \in \mathcal{T}_k^{\text{CF}}}\Big\}.$$

The k-th relay searches for the unique index $(\hat{w}_b, \hat{l}_{\mathcal{T}_k,b})$ by looking at two consecutive blocks b and $b+1$ such that $\mathcal{E}_k(\hat{w}_b, \hat{l}_{\mathcal{T}_k,b}) \in \mathcal{A}_\epsilon^n[VXX_kX_{\mathcal{T}_k}\hat{Z}_{\mathcal{T}_k}Z_k]$ and $\mathcal{E}_k(\hat{l}_{\mathcal{T}_k,b}) \in \mathcal{A}_\epsilon^n[VX_{\mathcal{T}_k}X_kZ_k]$. The error probability can be made arbitrary small provided that

$$
\begin{aligned}
&I(\hat{Z}_\mathcal{S}; Z_\mathcal{S}|VXX_kX_{\mathcal{T}_k}\hat{Z}_{\mathcal{S}^c}Z_k) < I(X_\mathcal{S}; Z_k|VX_kX_{\mathcal{S}^c}),\\
&R < I(X; \hat{Z}_{\mathcal{T}_k}Z_k|VX_kX_{\mathcal{T}_k}) + I(X_\mathcal{S}; Z_k|VX_kX_{\mathcal{S}^c})\\
&-I(\hat{Z}_\mathcal{S}; Z_\mathcal{S}|VX_kX_{\mathcal{T}_k}\hat{Z}_{\mathcal{S}^c}Z_k). \qquad (6)
\end{aligned}
$$

2) Decoding at the destination is done backwardly. After the last block, the decoder jointly searches for the unique indices $\left\{\hat{l}_{k,(B+2)} : k \in \mathcal{T}\right\}$ such that for all $b \in [B+3 : B+L]$, $\left\{\underline{x}_k(\hat{l}_{k,(B+1)}) : k \in \mathcal{T}^{\text{CF}}\right\}$ and $\left\{\underline{x}_k(1, \hat{l}_{k,(B+1)}) : k \in \mathcal{T}^{\text{DF}}\right\}$ are jointly typical with $\underline{x}(1,1)$, $\underline{v}(1)$ and $\underline{y}_1(b)$. The probability of error goes to zero as n goes to infinity provided that for all $\mathcal{S} \subseteq \mathcal{T}$:

$$
\sum_{k\in\mathcal{S}\cap\mathcal{T}^{\text{CF}}} I(\hat{Z}_k; Z_k|X_k) + \sum_{k\in\mathcal{S}\cap\mathcal{T}^{\text{DF}}} I(\hat{Z}_k; Z_k|X_kV) + \leq (L-2)I(X_\mathcal{S}; VXX_{\mathcal{S}^c}Y_1).
$$

3) After finding correctly $l_{k,(B+2)}$ for all $k \in \mathcal{T}$ and since $w_{(B+1)} = 1$, the destination decodes jointly the message and all the compression indices $(w_b, l_{\mathcal{T},(b+1)})$ for each $b = [1 : B]$ where $l_{\mathcal{T},b} = (l_{k,b})_{k\in\mathcal{T}}$. The decoding is done backwardly, assuming that $(w_{b+2}, l_{\mathcal{T},(b+2)})$ have been correctly decoded. Define the following sequence:

$$
\begin{aligned}
\mathcal{E}(\hat{w}_b, \hat{l}_{\mathcal{T},(b+1)}) = \Big\{&\underline{x}(\hat{w}_b, w_{(b+2)}), \underline{v}(\hat{w}_b), \underline{y}_1(b+2),\\
&\left(\underline{x}_k(\hat{l}_{k,(b+1)}), \hat{\underline{z}}_k(\hat{l}_{k,(b+1)}, l_{k,(b+2)})\right)_{k\in\mathcal{T}^{\text{CF}}},\\
&\left(\underline{x}_k(\hat{w}_b, \hat{l}_{k,(b+1)}), \hat{\underline{z}}_k(\hat{w}_b, \hat{l}_{k,(b+1)}, l_{k,(b+2)})\right)_{k\in\mathcal{T}^{\text{CF}}}\Big\}
\end{aligned}
$$

The destination finds the unique pair of indices $(\hat{w}_b, \hat{l}_{\mathcal{T},(b+1)})$ such that $\mathcal{E}(\hat{w}_b, \hat{l}_{\mathcal{T},(b+1)}) \in \mathcal{A}_\epsilon^n[VXX_\mathcal{T}\hat{Z}_\mathcal{T}Y_1]$. The probability of error goes to zero as $n \to \infty$, similar to [10] provided that

$$
\begin{aligned}
&I(X_\mathcal{S}; \hat{Z}_{\mathcal{S}^c}Y_1|XX_{\mathcal{S}^c}V) - I(\hat{Z}_\mathcal{S}; Z_\mathcal{S}|VXX_\mathcal{T}\hat{Z}_{\mathcal{S}^c}Y_1) > 0\\
&R < I(XX_\mathcal{S}; \hat{Z}_{\mathcal{S}^c}Y_1|X_{\mathcal{S}^c}) - I(\hat{Z}_\mathcal{S}; Z_\mathcal{S}|XX_\mathcal{T}\hat{Z}_{\mathcal{S}^c}Y_1)
\end{aligned}
$$

By choosing finite L but large enough, the previous inequalities along with (6) prove Theorem 1, where the rate is achieved by letting (B, n) tend to infinity. At the end, a time sharing random variable Q can be added.

V. Constant Gap for Gaussian Networks

In this section, we study constant gaps for AWGN networks. Before proceeding to the general case, we first consider the gap of DF rate for the single AWGN relay channel. CF constant gap follows the same line as [5]. Consider the Gaussian relay channel with average power inputs $\mathbb{E}[X^2] \leq P$, $\mathbb{E}[X_1^2] \leq P$ and channel outputs defined as follows:

$$
\begin{cases} Y_1 = h_3X + h_2X_1 + \mathcal{N}_1 \\ Z_1 = h_1X + \mathcal{N}_2, \hat{Z}_1 = Z_1 + \hat{\mathcal{N}}_2. \end{cases} \qquad (7)
$$

It is easy to check that the second term in the DF rate, corresponding to namely MAC part of the channel, appears untouched in the cut-set bound (CB) and the gap is zero between the term for MAC part and thus only the difference between first two terms affects the gap. This can be bounded as follows (β is the correlation coefficient):

$$
\begin{aligned}
\Delta(\text{CB}, \text{DF}) = &\frac{1}{2}\log\left(1 + \frac{\beta v_1 P + \beta v_3 P}{N}\right)\\
&- \frac{1}{2}\log\left(1 + \frac{\beta v_1 P}{N}\right) \leq \frac{1}{2}\log\left(1 + \frac{v_3}{v_1}\right).
\end{aligned}
$$

Observe that, unlike NNC, the gap for DF cannot be independent of the channel gains v_1 and v_3.

Now we move to the unicast multiple relay network. To this end, consider an AWGN Gaussian network with N relays, single source and destination of $N+2$ nodes. The relays are indexed as usual with index in the set $\mathcal{N} = \{1, \ldots, N\}$. For simplicity, we associate the source with the index 0, i.e. $X_0 = X$ and the destination with the index $N+1$. Therefore, there is bijection from the set of nodes to the set $\{0, 1, \ldots, N, N+1\}$. The transmitters' set is denoted by $\mathcal{M} = \{0, 1, \ldots, N\}$ and the receivers' set is denoted by $\mathcal{D} = \{1, \ldots, N, N+1\}$. By $\{g_{ij}\}$ we denote the channel gains from the i-th node to the node j. The noise at the node j is denoted by $\mathcal{N}_j$ which assumed to zero mean of unit variance complex Gaussian random variable. The input and output relation are defined as follows:

$$
\begin{cases} Y(\mathcal{D}) = G(\mathcal{D}, \mathcal{T})X(\mathcal{M}) + N(\mathcal{D}) \\ \hat{Z}(\mathcal{N}) = Z(\mathcal{N}) + \hat{N}(\mathcal{N}) \end{cases} \qquad (8)
$$

where $Y(\mathcal{D}) = [Z_1 \ldots Z_N Y_1]^T$. $G(\mathcal{D}, \mathcal{T})$ is the channel gain matrix where evidently $g_{ii} = 0$. Let $G(\mathcal{S}_1, \mathcal{S}_2)$ denote the set $[g_{ij} : i \in \mathcal{S}_1, j \in \mathcal{S}_2]$ and similarly for any A, $A(\mathcal{S}_1) = [a_i : i \in \mathcal{S}_1]$. All nodes transmit with power less or equal than P. By $\hat{N}(\mathcal{S})$ we denote the compression noise vector which is assumed to be Gaussian with unit variance. The covariance matrix of the channel inputs is $K(\mathcal{S}) = [P\rho_{ij}]$ for $i, j \in \mathcal{S}$.

To evaluate the constant gap for MNNC from Theorem 1, we assume $\mathcal{T} = \mathcal{N}$ with all inputs chosen as Gaussian. Given a set of channel gains, assume all relays in $\mathcal{V}^c$ use DF while those in $\mathcal{V}$ use CF. The constant gap for the conventional NNC has been evaluated in [5],

$$
\begin{aligned}
\Delta(\text{CB}, \text{NNC}) &\triangleq \max_{\mathcal{S}\subseteq\mathcal{N}}\left(\frac{|\mathcal{S}|}{2} + \frac{\min\{|\mathcal{S}|, |\mathcal{S}^c|\}}{2}\log(2|\mathcal{S}|)\right)\\
&\leq \frac{N+2}{4}\log(4N+4). \qquad (9)
\end{aligned}
$$

Proposition 1 (constant gap for MNNC): For unicast multiple relay networks, if the source-relay channel is good enough for DF relays, the constant gap can be stated as follows:

$$
\begin{aligned}
&\Delta(\text{CB}, \text{MNNC}) \triangleq \max_{\mathcal{N}-\mathcal{V}\subseteq\mathcal{S}\subseteq\mathcal{N}}\\
&\left[\frac{|\mathcal{S}|}{2} + \frac{1 + \min\{|\mathcal{S}|, |\mathcal{S}^c|\}}{2}\log\left(4(|\mathcal{V}| - |\mathcal{S}^c|)\right)\right].
\end{aligned}
$$

When only one relay is using DF, the gap is bounded by $\frac{N+2}{4}\log(4N-4)$ which is clearly less than NNC scheme.

To prove the proposition, we first find an upper bound on the cut-set bound using the following lemma:

Lemma 1: Suppose that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

are positive definite matrices then $2B \succeq A$.

To prove the lemma, it is enough to see that $2B - A$ is positive definite. Now we move to bounding cut-set bound. Suppose that $\mathcal{S}^{c*} = \mathcal{S}^c \cup \{N+1\}$, $\mathcal{V}^{c*} = \mathcal{V}^c \cup \{0\}$ and $\mathcal{S}_{\text{CF}} = \mathcal{S} \cap \mathcal{V}$ denoting the CF relays. The cut-set bound reads as

$$I(XX_{\mathcal{S}}; Z_{\mathcal{S}^c} Y_1 | X_{\mathcal{S}^c}) = \frac{1}{2} \log \Bigg(\left| I(\mathcal{S}^{c*}) + G \begin{bmatrix} K(\mathcal{V}^{c*}) & K(\mathcal{V}^{c*}, \mathcal{S}_{\text{CF}}) \\ K(\mathcal{S}_{\text{CF}}, \mathcal{V}^{c*}) & K(\mathcal{S}_{\text{CF}}) \end{bmatrix} G^T \right| \Bigg).$$

Now the determinant can be bounded as follows:

$$\left| I(\mathcal{S}^{c*}) + G \begin{bmatrix} K(\mathcal{V}^{c*}) & K(\mathcal{V}^{c*}, \mathcal{S}_{\text{CF}}) \\ K(\mathcal{S}_{\text{CF}}, \mathcal{V}^{c*}) & K(\mathcal{S}_{\text{CF}}) \end{bmatrix} G^T \right| \overset{(a)}{\leq} \left| 2I(\mathcal{S}^{c*}) + 2|\mathcal{S}_{\text{CF}}| G \begin{bmatrix} K(\mathcal{V}^{c*}) & 0 \\ 0 & PI(\mathcal{S}_{\text{CF}}) \end{bmatrix} G^T \right| \tag{10}$$

where (a) comes from Lemma 1, and from $\text{Tr}(\mathcal{S}_{\text{CF}}) I(\mathcal{S}_{\text{CF}}) \succeq K(\mathcal{S}_{\text{CF}})$. By rewriting (10) similar too [5], we obtain

$$R \leq \frac{1}{2} \log \left(\left| I(\mathcal{S}^{c*}) + \frac{1}{2} G \begin{bmatrix} K(\mathcal{V}^{c*}) & 0 \\ 0 & PI(\mathcal{S}_{\text{CF}}) \end{bmatrix} G^T \right| \right) + \frac{\min\{|\mathcal{S}^c|+1, |\mathcal{S}|+1\}}{2} \log \left((4|\mathcal{S}_{\text{CF}}|) \right). \tag{11}$$

The term $I(\hat{Z}_{\mathcal{S}}; Z_{\mathcal{S}} | X X_{\mathcal{N}} \hat{Z}_{\mathcal{S}^c} Y_1)$ is lower bounded by $I(\hat{Z}_{\mathcal{S}}; Z_{\mathcal{S}} | X X_{\mathcal{N}}) = |\mathcal{S}|/2$. Notice in the first part of $R_{\mathcal{N}}(\mathcal{S})$, unlike NNC, the relays from $\mathcal{V}^c$ are not independent and hence

$$I(XX_{\mathcal{S}}; \hat{Z}_{\mathcal{S}^c} Y_1 | X_{\mathcal{S}^c}) \geq I(XX_{\mathcal{S}}; \hat{Z}_{\mathcal{S}^c} \hat{Y}_1 | X_{\mathcal{S}^c}) = \frac{1}{2} \log \left(\left| I(\mathcal{S}^{c*}) + \frac{1}{2} G \begin{bmatrix} K(\mathcal{V}^{c*}) & 0 \\ 0 & PI(\mathcal{S}_{\text{CF}}) \end{bmatrix} G^T \right| \right)$$

where the matrix $K(\mathcal{V}^{c*})$, is the one that maximizes CB. So the gap between $R_{\mathcal{N}}(\mathcal{S})$ and its corresponding CB becomes:

$$\Delta_1(\text{CB,MNNC}) \triangleq \max_{\mathcal{N} - \mathcal{V} \subseteq \mathcal{S} \subseteq \mathcal{N}} \left[\frac{|\mathcal{S}|}{2} + \frac{\min\{|\mathcal{S}^c|+1, |\mathcal{S}|+1\}}{2} \log \left((4|\mathcal{S}_{\text{CF}}|) \right) \right]. \tag{12}$$

In order to bound the other part of the rate related to DF relays, note that Y_1 is absent in the rate expression as for DF in the single AWGN relay channel, while it is always present in the CB. Thus, the gap between the rate and the CB, no matter how good we manage to bound it, will depend on the channel gains between Y_1 and inputs. For sake of clarity, let us assume that each DF relay is decoding the source message alone, i.e., without using the compression index of other relays yielding $\mathcal{T}_k = \emptyset$. Given this choice, the rate $R^{(k)}_{\mathcal{T}_k}$ is reduced to $R^{(k)}_{\text{DF}} = I(X; Z_k | V X_k)$. First, observe that

$$Z_k = g_{0k} X + \sum_{i \in \mathcal{V}} g_{ik} X_i + \sum_{i \in \mathcal{V}^c} g_{ik} X_i + \mathcal{N}_k,$$

where similarly $\mathcal{V}$ is the set of users using CF. From which the mutual information can be stated as follows:

$$I(X; Z_k | V X_k) = \frac{1}{2} \log \left[1 + \frac{|g_{0k}|^2 (1 - \rho_{0k}^2) P}{\sum_{i \in \mathcal{V}} |g_{ik}|^2 P + \sum_{i \in \mathcal{V}^c} |g_{ik}|^2 (1 - \rho_{ik}^2) P + 1} \right].$$

The cut-set bound corresponding to this rate is calculated as:

$$I(XX_{\mathcal{N} \backslash \{k\}}; Z_k Y_1 | X_k) = \frac{1}{2} \log \left(\left| I(2) + G K(\mathcal{M} \backslash \{k\}) G^T \right| \right).$$

As it is expected, this gap, say $\Delta_2(\text{CB}, \text{MNNC})$, is not independent of the channel gains. The final gap $\Delta(\text{CB}, \text{MNNC})$ would be the maximum of $\Delta_1(\text{CB}, \text{MNNC})$ and $\Delta_2(\text{CB}, \text{MNNC})$ and it is dependent on the channel gains. However, if DF relays are chosen correctly then they should not contribute to increasing the gap and the dominant term in the gap is $\Delta_1(\text{CB}, \text{MNNC})$ which is independent of the channel gain. Now the question is to find the proper condition for choosing the DF relays. One solution is to choose the relays with g_{0k} enough large compared to other gains which indicates that the channel quality of source-relay is very good.

ACKNOWLEDGMENT

The authors are grateful to Prof. Abbas El Gamal for his valuable advice at the early stage of this work. They are also grateful to Prof. Gerhard Kramer for valuable comments and suggestions.

REFERENCES

[1] T. Cover and A. El Gamal, "Capacity theorems for the relay channel," *Information Theory, IEEE Trans. on*, vol. IT-25, pp. 572–584, 1979.

[2] Y. Liang and G. Kramer, "Rate regions for relay broadcast channels," *Information Theory, IEEE Transactions on*, vol. 53, no. 10, pp. 3517–3535, Oct. 2007.

[3] G. Kramer, M. Gastpar, and P. Gupta, "Cooperative strategies and capacity theorems for relay networks," *Information Theory, IEEE Transactions on*, vol. 51, no. 9, pp. 3037–3063, Sept. 2005.

[4] A. El Gamal, M. Mohseni, and S. Zahedi, "Bounds on capacity and minimum energy-per-bit for AWGN relay channels," *IEEE Trans. Information Theory*, vol. 52, no. 4, pp. 1545–1561, Apr. 2006.

[5] S. H. Lim, Y.-H. Kim, A. El Gamal, and S.-Y. Chung, "Noisy network coding," *Information Theory, IEEE Transactions on*, vol. 57, no. 5, pp. 3132 –3152, may 2011.

[6] A. Dana, R. Gowaikar, R. Palanki, B. Hassibi, and M. Effros, "Capacity of wireless erasure networks," *Information Theory, IEEE Transactions on*, vol. 52, no. 3, pp. 789 –804, march 2006.

[7] A. Avestimehr, S. Diggavi, and D. Tse, "Wireless network information flow: A deterministic approach," *Information Theory, IEEE Transactions on*, vol. 57, no. 4, pp. 1872 –1905, april 2011.

[8] X. Wu and L.-L. Xie, "On the optimal compressions in the compress-and-forward relay schemes," *Arxiv:*, Feb. 2011. [Online]. Available: http://arxiv.org/abs/1009.5959

[9] J. Hou and G. Kramer, "Short message noisy network coding for multiple sources," in *Information Theory Proceedings (ISIT), 2012 IEEE International Symposium on*, july 2012, pp. 1677 –1681.

[10] A. Behboodi and P. Piantanida, "Selective coding strategy for unicast composite networks," in *Information Theory Proceedings (ISIT), 2012 IEEE International Symposium on*, 2012.

[11] J. Du, M. Xiao, M. Skoglund, and S. Shamai, "Short-message noisy network coding with partial source cooperation," in *Information Theory Workshop (ITW), 2012 IEEE*, sept. 2012, pp. 144 –147.