Oblivious Lattice Codes for Gaussian Relay Channels

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***Abstract*—Consider a slowly fading Gaussian relay channel where the source is not aware of channel state information (CSI) and the relay is only partially aware of CSI. As a single cooperation strategy, Decode-and-Forward (DF) or Compress-and-Forward (CF), is not the best for all channel states, the relay should be able to switch between them according to its CSI. However as the source cannot be aware of chosen cooperative strategy, it should use so called oblivious codes that perform equally well under different cooperative strategies. In this paper, we prove that doubly nested lattice codes is an oblivious code which can be used to achieve DF, CF or point-to-point rate in the relay channels. Using the oblivious lattice coding, the relay, according to its CSI, decides to use DF, CF or no cooperation at all. We show that inner bound on outage probability is significantly improved if the oblivious lattice code is used with selective coding strategy at the relay.**

I. Introduction

In the original formulation for the problem of single relay channel [1], [2], the channel was assumed to be a conditional probability known to both encoders and decoders. Nonetheless, frequently in the practical communication system, the conditional probability corresponding to the channel variates due to the effect of a set of parameters such as fading and mobility. As the source does not have any observation of the channel, the information about these parameters is usually not available at the source and it may be only partly available at the relay. A common model for this situation is channels with state where the state is chosen before the communication and does not change during the communication however the source is not aware of the realization of the state. This case corresponds to quasi-static channel or slowly fading channels. The capacity of this class of channels has been studied in the literature using notions such as capacity-versus-outage for composite channels and the $\epsilon-$capacity of averaged channels [3].

In the static channels, the superposition coding is used at the source if the relay intends to use Decode-and-Forward (DF) while a conventional single user code is used at the source in Compress-and-Forward (CF) case. In quasi-static channels, neither DF nor CF can perform better than the other for all possible channel states. Sometimes, the relay should choose DF and sometimes CF. But the source, unaware of channel realization, cannot know this in advance. An oblivious code is the code that can be used at the source and performs well regardless of relay strategy. This enables the relay to adaptively change its cooperative strategies. The authors in [4], [5] proved that the superposition coding is an oblivious good which means that it can also be used to achieve the CF rate. The authors introduced selective coding strategy at the relay where they proved that it is always beneficial for the relay to use DF if it can decode the source message according to its own CIS. In this work, we argue whether the similar results obtained for random coding applies also for structural codes too. In other words, we show that there is a lattice code which can be used to achieve both CF rate and DF rate.

There are significant amount of research on structural codes and its application to different multi-user information theory problems [6]–[9]. Moreover the doubly nested lattice code has been shown to achieve DF rate [10], [11] while the CF rate is obtained using single nested lattice codes. Another interesting line of work is Compute-and-Forward (CompF) scheme [12] which is shown to outperform the DF and CF rates in some scenario. Therefore for general slowly fading multi-terminal networks, the relays should be able to switch between DF, CF and CompF according to channel condition. A first step toward this general selective coding strategy is to show that there is an oblivious lattice code which works well for single user case. We show that doubly nested lattice codes are oblivious codes that can be used to achieve the achievable rate of DF, CF and single user channels.

The paper is organized as follows. Next section is dedicated to the definitions and overview of lattice codes used in the literature. In the section III we discuss about oblivious nested lattice codes and prove that it achieves DF and CF rate both. In the next section, numerical results are provided for selective coding strategy at the relay where a significant improve in throughput is observed.

II. Main Definitions and Problem Statements

A. Relay Channels

Consider a set of relay channels $\left\{P_{Y_{2\theta}^n Y_{1\theta_r}^n | X^n X_{1\theta_r}^n}\right\}_{n=1}^{\infty}$ where the parameters $\theta = (\theta_d, \theta_r) \in \mathit{\Theta}$ represent the channel state information. The parameter $\theta_r \in \mathit{\Theta}_r$ and $\theta_d \in \mathit{\Theta}_d$ respectively refer to CSI of the relay output and the CSI of destination. The channel, i.e. θ, is chosen at the beginning of communication according to $\mathbb{P}_\theta$ and remains fixed during the rest of communication. Each channel is denoted by conditional PD $\left\{P_{Y_{2\theta} Y_{1\theta_r} | X X_{1\theta_r}} : \mathcal{X} \times \mathcal{X}_1 \longmapsto \mathcal{Y}_1 \times \mathcal{Y}_2\right\}$.

The channel parameters affecting relay and destination outputs $\theta = (\theta_r, \theta_d)$ are assumed to be unknown at the source,

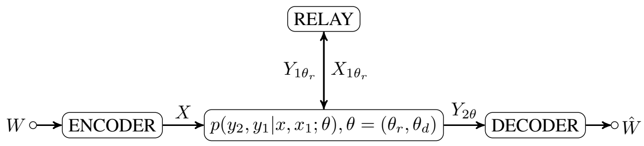

Fig. 1: Relay Channels with State

fully known at the destination and partly known θ_r at the relay end.

This channel in Fig. 1 belongs to a more general class of channels known as channel with state, which is in general a model for situations in which the channel variation is controlled by another random variable, called S. By changing S, the channel is changed. In this case the state is chosen before the communication starts and does not change during the communication however the encoder is not aware of the realization of the state. This case corresponds to quasi-static channel or slowly fading channels. The state S_i at the time i is equal to $\theta \in \Theta$ for all i's where Θ is set of all possible states. The compound capacity of this channel can turn out to be zero and therefore we have to consider probability distribution attributed to θ.

One option is to allow error for some states provided that the probability of those states does not exceed certain threshold. Hence, the encoder chooses a fixed rate and if the fading coefficient is such that the message cannot be correctly decoded, an outage is declared. The probability assigned to this event is called outage probability. For an arbitrary channel with states, the outage event is declared after n transmission with an outage identification function $\mathcal{I}_n$ [13] which is a function of channel states, defined as $\mathcal{I}_n : \Theta \to \{0, 1\}$. For this definition, the outage is declared when $I_n(\theta) = 0$ and therefore the outage probability is defined as:

$$P_{n,\text{out}} = \Pr(\mathcal{I}_n(\theta) = 0).$$

Definition 1 (Code and outage probability)**.** *A code-$\mathcal{C}(n, M_n)$ for the composite relay channel consists of:*

- *An encoder mapping $\{\varphi : \mathcal{M}_n \longmapsto \mathcal{X}^n\}$,*
- *A set of decoder mappings $\{\phi_\theta : \mathcal{Y}_2^n \longmapsto \mathcal{M}_n\}$, for all $\theta \in \Theta$*
- *A set of relay functions $\left\{f_{i,\theta_r} : \mathcal{Y}_1^{i-1} \longmapsto \mathcal{X}_1\right\}_{i=1}^n$, for some set of uniformly distributed message $W \in \mathcal{M}_n = \{1, \ldots, M_n\}$ and for all $\theta_r \in \Theta_r$. Note that only partial CSI at the relay is assumed (denoted by θ_r) which is mainly related to the source-relay link.*
- *Outage identification function $\mathcal{I}_n : \Theta \to \{0, 1\}$.*

A pair of rate-outage probability (R, ϵ) is said to be achievable, if there exists a code-$\mathcal{C}(n, M_n)$ such that:

$$\liminf_{n\to\infty} \frac{1}{n} \log M_n \geq R \text{ and } P_{n,out} \leq \epsilon \tag{1}$$

and

$$\limsup_{n\to\infty} \Pr\left\{\phi_\theta(Y_{2\theta}^n) \neq W \middle| I_n(\theta) = 1\right\} = 0. \tag{2}$$

For the example of slowly fading channels, the outage identification function is simply an indicator function, indicating whether the transmission rate is smaller than the capacity of the channel with realized fading coefficients. The capacity versus outage with outage probability ϵ, $C_{\text{out},\epsilon}$ is defined as the supremum of all achievable rates with maximum outage probability ϵ. Here we are concerned with the AWGN slowly fading relay channel where θ_r and θ_d is the fading coefficients of channels to the relay and to the destination, respectively.

B. Lattice Codes

In this part, we overview some of well known results on lattice codes that can be found in [6], [7]. Lattices are simply set of all integer linear combination of basis vectors in $\mathbb{R}^n$ which is $\Lambda = \{Gx : x \in \mathbb{Z}^n\}$ where $G = [\mathbf{g}_1 \ldots \mathbf{g}_n]$ with $\mathbf{g}_i \in \mathbb{R}^n$. They are considered as generalization of linear block codes and also the idea of constellation. Lattices can be used for quantization. For instance each point in $\mathbb{R}^d$ can be associated with the nearest lattice points which results in nearest neighbor lattice quantizer $Q(x) = \arg\min_{\lambda\in\Lambda} \|x - \lambda\|$. The fundamental Voronoi region of lattice Λ is defined as the set of all points that are quantized to zero, namely $\mathcal{V} = \{x : Q(x) = 0\}$. The modulo-$\Lambda$ operation is defined as $x \bmod_{\mathcal{V}} \Lambda = x - Q(x)$. The fundamental Voronoi region is important because it corresponds to nearest neighbor decoding/quantization region. Probability of error in each case is related to the probability that the received/quantized vector falls outside the Voronoi region around the original vector. The volume of Voronoi region V_Λ is given by $|\det(G)|$. Energy per dimension, i.e. mean squared quantization or second moment of lattice is defined as the power per dimension of a random variable uniformly distributed over $\mathcal{V}$:

$$\sigma_\Lambda^2 = \frac{1}{n}\mathbb{E}(\|X\|^2) = \int_{\mathcal{V}} \|X\|^2 \frac{1}{nV_\Lambda} dX.$$

Normalized second moment (NSM) is defined as $G(\Lambda) = \frac{\sigma_\Lambda^2}{V_\Lambda^{\frac{2}{n}}}$. NSM is invariant under scaling and orthonormal transformation. The value of NSM for a ball of radius R, in n dimensional space tends to $\frac{1}{2\pi e}$ as n goes to infinity. Note that the entropy of a uniformly distributed random variable inside $\mathcal{V}$ is equal to $\frac{1}{2}\log(\frac{\sigma_\Lambda^2}{G(\Lambda)})$. Intuitively, NSM replaces $\frac{1}{2\pi e}$ in the formula for the entropy of a Gaussian random variable with variance σ^2. We are interested in those Λ_n for which $G(\Lambda_n)$ is minimum and therefore the entropy of a uniform RV over $\mathcal{V}$ is maximum. This amounts to having $\lim_{n\to\infty} G(\Lambda^n) = \frac{1}{2\pi e}$. These are called *good for mean-squared error quantization*. One class of lattices, satisfying this property, are Rogers good lattices. A sequence of n-dimensional lattices Λ_n is Rogers good if:

$$\lim_{n\to\infty} \frac{r_{\text{cov}}^n}{r_{\text{eff}}^n} = 1.$$

r_{cov}^n is the covering radius, the radius of smallest ball including the fundamental Voronoi region and r_{eff}^n is the effective radius of lattice, the radius of a ball with the same volume as the fundamental Voronoi region. If we look for good lattices for

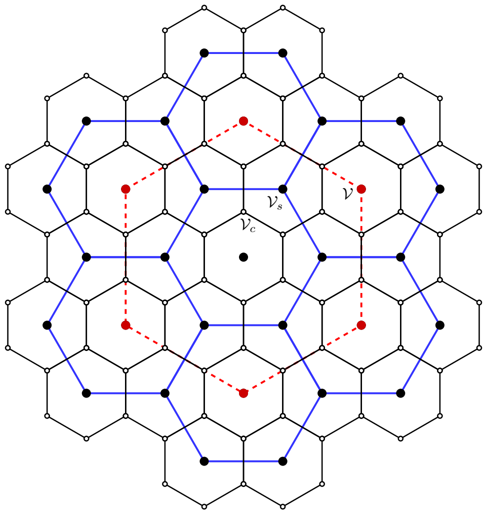

Fig. 2: Doubly Nested Lattice Code

channel coding over AWGN channel, those lattices should have a fundamental Voronoi region such that the probability of error, i.e. probability of noise falling outside that region, is exponentially diminishing. A specific sense of such goodness is captured by *Poltyrev good* lattices. A sequence of n-dimensional lattices Λ^n is *Poltyrev good*, if for n-dimensional i.i.d. Gaussian noise Z^n with power P_N, we have:

$$\Pr(Z^n \notin \mathcal{V}_{\Lambda^n}) \leq e^{-n(E_P(\mu)-o_n(1))}.$$

μ is the volume-to-noise ratio (VNR) which is defined as:

$$\mu = \frac{(V_{\Lambda^n})^{\frac{2}{n}}}{2\pi e P_N} + o_n(1) = \frac{\sigma^2_{\Lambda^n}}{P_N}\frac{1}{2\pi e G(\Lambda^n)} + o_n(1)$$

and $E_P(\mu)$ is the Poltyrev exponent and it is defined as:

$$E_P(\mu) = \begin{cases} \frac{1}{2}[(\mu-1)-\log\mu] & 1<\mu<2 \\ \frac{1}{2}\log\frac{e\mu}{4} & 2\leq\mu\leq 4 \\ \frac{\mu}{8} & \mu\geq 4 \end{cases}$$

Note that interestingly enough, the effective power of lattice constellation is $\frac{(V_{\Lambda^n})^{\frac{2}{n}}}{2\pi e}$ which is only equal to the energy per dimension of lattice for other kinds of lattices. Poltyrev good lattices guarantee only the vanishing error probability for good choice of VNR. This good choice corresponds to the choice of $\mu > 1$.

The idea of constellation is mimicked by using two lattices. The fundamental Voronoi region of the first lattice, which is the coarser lattice, guarantees the power constraint of the code while the finer lattice provides the constellation points inside the Voronoi region of the coarser lattice. This structure is called nested lattice codes with $\Lambda \subset \Lambda_c$ where the constellation points are chosen from $\mathcal{V} \cap \Lambda_c$. The rate of the nested lattice code is then defined as $R = \frac{1}{n}\log\frac{V_\Lambda}{V_{\Lambda_c}}$. The capacity of AWGN channels can be achieved using nested lattice with similar guidelines from random coding argument. Codewords are designed to have Gaussian distribution to guarantee maximum entropy. To do this, after selecting the constellation point, we add U, a uniformly distributed RV over $\mathcal{V}$, known as dither, to the lattice point. The following lemma guarantees that the channel input has also uniform distribution over $\mathcal{V}$.

Lemma 1. *(Crypto lemma) For any random variable X defined over Voronoi region $\mathcal{V}$, and an independent uniformly distributed random variable U over $\mathcal{V}$. Then $Y = X + U \bmod_{\mathcal{V}} \Lambda$ is also uniformly distributed over $\mathcal{V}$ and it is statistically independent of X.*

If we choose the coarser lattice to be Rogers good, then the channel input will asymptoticly have Gaussian-like entropy. The coarser lattice should also be Poltyrev good. With this choice, the coarser lattice provides us with proper input distribution and power constraint. The finer lattice should only be Poltyrev good to be good for AWGN decoding. At the end, the original AWGN channel is converted to the following equivalent modulo-lattice channel.

Lemma 2. *(Inflated lattice lemma) Consider the AWGN channel, namely $Y = X + Z$. If $t \in \mathcal{V}$ is the codeword and the dither U, one can send $X_t = [t - U] \bmod_{\mathcal{V}} \Lambda$ and then find $Y' = [\alpha Y + U] \bmod_{\mathcal{V}} \Lambda$. The channel is then represented as $Y = [t + Z'] \bmod_{\mathcal{V}} \Lambda$ where $Z' = [\alpha Z + (1-\alpha)U] \bmod_{\mathcal{V}} \Lambda$.*

The parameter α is chosen as MMSE coefficient and its role is to inflate the Voronoi region to include the noise sphere.

III. Oblivious Nested Lattice Coding

To extend the previous setting to relays, we have to extend the nested lattice codes to nested lattice chains. Doubly nested lattice code, Figure 2, is defined by three lattices $\Lambda, \Lambda_s, \Lambda_c$, satisfying $\Lambda \subset \Lambda_s \subset \Lambda_c$. Doubly nested lattice codes can be employed to achieve Decode-and-Forward region either by using list decoding [10] or by using cloud-satellite codes [11]. We choose the later code however the former can be equally used. The original constellation points belong to $\mathcal{V} \cap \Lambda_c$. Each $t \in \mathcal{V} \cap \Lambda_c$ can be written as $t_0 + t_1$. t_0 belongs to $\mathcal{V} \cap \Lambda_s$ and presents the coarser constellation. Each $t_0 + \mathcal{V}_s$ contains a subset of original constellation. t_1 provides resolution information to find the exact $t_0 + \mathcal{V}_s$ and it is chosen from $\mathcal{V}_s \cap \Lambda_c$. The structure is reminiscent of superposition coding with cloud centers and satellites. The rate of lattices are defined similarly to be $R = \frac{1}{n}\log\frac{V_\Lambda}{V_{\Lambda_c}}$ and $R_0 = \frac{1}{n}\log\frac{V_{\Lambda_s}}{V_{\Lambda_c}}$ and $R_1 = R - R_0$. Λ_c needs to be only Poltyrev good while the other two lattices should be both Rogers and Poltyrev good. The following theorem states the achievability of DF rate with this lattice code.

Theorem 1. *([10], [11]) Doubly nested lattice code can be used to achieve DF achievable rate region for AWGN relay channel.*

Proof. We use backward decoding for the proof nevertheless the idea follows the similar line as in [11]. In any case we present outline of the proof, as it is used for the next theorems.

Consider a doubly nested lattice code with $\Lambda, \Lambda_s, \Lambda_c$, satisfying $\Lambda \subset \Lambda_s \subset \Lambda_c$ with Λ satisfying unit power. The original constellation points belong to $\mathcal{V} \cap \Lambda_c$. Each $t \in \mathcal{V} \cap \Lambda_c$ can be written as $t_0 + t_1$. t_1 belongs to $\mathcal{V} \cap \Lambda_s$ and presents the coarser constellation. Each $t_1 + \mathcal{V}_s$ contains a subset of original constellation. t_0 provides resolution information to find the exact $t_1 + \mathcal{V}_s$ and it is chosen from $\mathcal{V}_s \cap \Lambda_c$. The structure is reminiscent of superposition coding with cloud centers and satellites. The rate of lattices are defined similarly to be $R = \frac{1}{n} \log \frac{V_\Lambda}{V_{\Lambda_c}}$ and $R_0 = \frac{1}{n} \log \frac{V_{\Lambda_s}}{V_{\Lambda_c}}$ and $R_1 = R - R_0$. Λ_c needs to be only Poltyrev good while the other two lattices should be both Rogers and Poltyrev good.

2^{nR} messages are associated to constellation points t in $\mathcal{V} \cap \Lambda_c$. Each point is written as $t_0 + t_1$ which is similar to writing t as sum of two messages with rate R_0 and R_1. At the block i, the source sends $t(i)$ with $t_0(i-1)$ after adding dithers U, U_s as follows:

$$\begin{aligned} X(i) = &\sqrt{\overline{\alpha} P}([t_1(i) + t_0(i) - U] \mod {}_{\mathcal{V}}\Lambda) \\ &+ \sqrt{\alpha P}([t_0(i-1) - U_s] \mod {}_{\mathcal{V}_s}\Lambda_s^*) \end{aligned}$$

where Λ_s^*, Λ_c^* are scaled lattices to satisfy unit power. The relay observes $Y_1(i) = X(i) + Z_1(i)$ and then it decodes $t(i)$ after the block i, knowing $t_0(i-1)$. Assuming same noise N, the condition for correct decoding is as follows:

$$R \leq \frac{1}{2} \log(1 + \frac{\overline{\alpha} P}{N}). \tag{3}$$

In block i, it forwards $t_0(i-1)$ as follows:

$$X_1(i) = \sqrt{P_1}([t_0(i-1) - U_s] \mod {}_{\mathcal{V}_s}\Lambda_s^*).$$

The destination therefore receives the following:

$$\begin{aligned} Y_2(i) = &Z_2(i) + \sqrt{\overline{\alpha} P}([t_1(i) + t_0(i) - U] \mod {}_{\mathcal{V}}\Lambda) \\ &+ (\sqrt{\alpha P} + \sqrt{P_1})([t_0(i-1) - U_s] \mod {}_{\mathcal{V}_s}\Lambda_s^*). \end{aligned}$$

Decoding starts backwardly. The destination decodes first $t_0(b)$ at the block $b+1$, and then it decodes $t_1(b)$ from the block b, assuming that all $t_1(b+j), t_0(b+j)$ are known and their terms are subtracted:

$$\begin{aligned} Y_2(b+1) =&Z_2(b+1) \\ &+ (\sqrt{\alpha P} + \sqrt{P_1})([t_0(b) - U_s] \mod {}_{\mathcal{V}_s}\Lambda_s^*). \end{aligned}$$

$t_0(b)$ is decoded correctly, after using inflated lattice lemma with the choice of MMSE estimator, if $R_0 \leq \frac{1}{2} \log(1 + \frac{\sqrt{\alpha P P_1} + P_1 + \alpha P}{N})$. Knowing $t_0(b)$ we subtract it from $Y_2(b)$ and we have:

$$\begin{aligned} \tilde{Y}_2(b) = &Z_2(b) + \sqrt{\overline{\alpha} P}[t_1(b) - U] \mod {}_{\mathcal{V}}\Lambda \\ &+ (\sqrt{\alpha P} + \sqrt{P_1})[t_0(b-1) - U_s] \mod {}_{\mathcal{V}_s}\Lambda_s^*. \end{aligned}$$

Note that all terms pertaining to $t_0(b-1)$ are uniformly distributed according to Crypto lemma and act like noise for $t_1(b)$, which is then decoded correctly if:

$$R_1 \leq \frac{1}{2} \log(1 + \frac{\overline{\alpha} P}{\sqrt{\alpha P P_1} + P_1 + \alpha P + N}).$$

As a result, the final rate R is achievable if:

$$R = R_1 + R_0 \leq \frac{1}{2} \log(1 + \frac{\sqrt{\alpha P P_1} + P_1 + P}{N}). \tag{4}$$

The DF rate is obtained using (3) and (4). ∎

In the next theorem, we prove that that doubly nested lattice codes can achieve the capacity of single user (SU) channel.

Theorem 2. *Doubly nested lattice codes with $\Lambda, \Lambda_s, \Lambda_c$, satisfying $\Lambda \subset \Lambda_s \subset \Lambda_c$ with Λ satisfying unit power can be used to achieve the capacity of single user channel namely* $\frac{1}{2} \log(1 + \frac{P}{N})$.

Proof. As we have seen in the proof of previous theorem, the destination can use backward decoding to recover all messages even if the relay is not present. This is equivalent to putting the relay power to zero. ∎

An important consequence of Theorem 2 is that if the source uses doubly nested lattice code with the hope that the relay is helping the decoding process and yet the relay is not present, the communication does not suffer from this absence and the destination can still achieve the single user channel capacity. In slowly fading channels, this means that the relay can switch between transmission and silence in function of channel condition and the message can still be decoded. In other words, doubly nested lattice code is an oblivious code.

Another question is whether doubly nested lattice code can also be used for CF relaying. If this is the case, the relay can switch between DF and CF without any loss in performance and without the need for the source to know which strategy is used. The next theorem present results in this direction.

Theorem 3. *(Doubly nested lattice codes with $\Lambda, \Lambda_s, \Lambda_c$, satisfying $\Lambda \subset \Lambda_s \subset \Lambda_c$ with Λ satisfying unit power can achieve the CF rate of the relay channels.*

Proof. The encoder is supposed to use the same doubly nested lattice code as introduced before. On the other hand, the relay uses CF strategy which means that it compresses its observation with a given distortion and transmits it to the destination. A modified version of lattice coding for Wyner-Ziv has been used to achieve Compress-and-Forward rate in [10] where the relay uses a single nested lattice code to quantize its channel observation. The finer lattice guarantees that the quantization distortion does not exceed $\hat{N}_1$ and the coarser lattice is designed in a way that its Voronoi region contains the relay observation Y_1 after scaling, dithering and added side information. The quantized version is transmitted through the channel and is decoded using the side information. The details are exactly same as [10] and we omit the proof. The distortion D should satisfy at the end the following constraint:

$$\hat{N}_1 \geq N_1 \frac{P(\frac{1}{N_1} + \frac{1}{N_2}) + 1}{\frac{P_1}{N_2}}.$$

The decoding starts backwardly and we assume that the destination starts at the block $B+2$ to decode $\hat{Y}_1(B+1)$ and continue backwardly to decode other quantized version.

After the correct decoding of $\hat{Y}_1$, maximum ratio combining (MRC) is used to combine it with Y_2 to decode the proper messages. Following lemma provides slightly more general results on MRC.

Lemma 3. *Consider a SIMO AWGN channel defined by $Y_i = X + U + Z_i$ for $i \in \{1, 2, \dots, n\}$ where the same X and U are transmitted over all channels with power P and P_1. Then the maximum ratio combining achieves the capacity of this channel with the ratio α_i of Y_i chosen as $\alpha_i = \frac{\prod_{j\neq i} N_j}{\sum_i \prod_{j\neq i} N_j}$. The capacity of this channel is equal to $C = \frac{1}{2}\log(1+\frac{P}{P_1+\tilde{N}})$ where $\tilde{N} = \left(\frac{1}{N_1} + \cdots + \frac{P}{N_n}\right)^{-1}$.*

Proof. The proof of theorem follows from direct calculation. The capacity of this channel is known as $I(X; Y_1 \dots Y_n)$, which is obtained by decoding X^n using all Y_i^n and is obtained as $C = \frac{1}{2}\log(1+\frac{P}{P_1+\tilde{N}})$. It only remains to show that the same can be achieved through MRC. This can be verified by choosing α_i as in Lemma (3) and using $\tilde{Y} = \sum_i \alpha_i Y_i$ for decoding. Note that when $P_1 = 0$, the capacity is simply equal to $\frac{1}{2}\log(1 + \frac{P}{N_1} + \cdots + \frac{P}{N_n})$. ∎

MRC can be directly applied to single nested lattice codes to achieve CF rate as in [10]. Using MRC at the decoder is different here because of doubly nested lattice codes. The decoding is done in two steps. The decoder starts backwardly by decoding first $t_0(b)$ at the block $b+1$ using both $\hat{Y}_1$ and Y_2. Assuming that all previous messages have been correctly decoded, :

$$\begin{aligned} Y_2(b+1) =& Z_2(b+1) \\ &+ (\sqrt{\alpha P})([t_0(b) - U_s] \mod {\mathcal{V}_s}\Lambda_s^*) \\ \hat{Y}_1(b+1) =& \hat{Z}_1(b+1) + Z_1(b+1) \\ &+ (\sqrt{\alpha P})[t_0(b) - U_s] \mod {\mathcal{V}_s}\Lambda_s^*. \end{aligned}$$

Using Lemma 3, $t_0(b)$ is decoded correctly, after using inflated lattice lemma with proper scaling by MMSE estimator, if $R_0 \leq \frac{1}{2}\log(1 + \frac{\alpha P}{\tilde{N}})$. where $\frac{1}{\tilde{N}} = \frac{1}{N_1+\hat{N}_1} + \frac{1}{N_2}$. The next step is to decode $t_1(b)$ from the block b. Knowing $t_0(b)$ we subtract it from $Y_2(b)$ and we have:

$$\begin{aligned} \tilde{Y}_2(b) &= Z_2(b) + \sqrt{\overline{\alpha} P}[t_1(b) - U] \mod {\mathcal{V}}\Lambda \\ &\quad + (\sqrt{\alpha P})[t_0(b-1) - U_s] \mod {\mathcal{V}_s}\Lambda_s^* \\ \hat{Y}_1(b) &= \hat{Z}_1(b) + Z_1(b) + \sqrt{\overline{\alpha} P}[t_1(b) - U] \mod {\mathcal{V}}\Lambda \\ &\quad + (\sqrt{\alpha P})[t_0(b-1) - U_s] \mod {\mathcal{V}_s}\Lambda_s^*. \end{aligned}$$

Here we have a situation similar to Lemma 3, where all terms pertaining to $t_0(b-1)$ act like the dither U_s. Using the lemma, the decoding is successful if $R_1 \leq \frac{1}{2}\log(1+\frac{\overline{\alpha} P}{\alpha P+\tilde{N}})$. It is easy to see that by correctly decoding of $t_0(b)$ and $t_1(b)$, the sum rate of $R_0 + R_1$ is achievable which is as follows:

$$R_0 + R_1 \leq \frac{1}{2}\log(1 + \frac{P}{N_1 + \hat{N}_1} + \frac{P}{N_2}).$$

Therefore we have proved that doubly nested lattices are oblivious codes for cooperation over relay channels. ∎

The set of results that we presented at this section implies that doubly nested lattice codes can be used instead of single nested lattice codes to achieve single user channel capacity and CF achievable rate. A similar result can be proved for multiple nested lattice codes. The significance of this result as we will see in the next section is that the source can use the doubly nested lattice code without knowing whether the relay is using DF or CF or not cooperating at all. This is very useful in slowly fading channels where the source does not know about the quality of source-relay channel and therefore should pick the best one.

IV. Selective Coding Strategy

As we have shown that doubly nested lattice code is an oblivious code, in this section, we use it for slowly fading relay channels. Consider the single user Gaussian channel:

$$Y_2 = H_1 X + H_3 X_1 + Z_2, Y_1 = H_2 X + Z_1,$$

where Z_1 and Z_2 are the additive noises of unit variance, i.i.d. circularly symmetric complex Gaussian RVs with zero-mean and unit variance. In addition to this, (H_1, H_2, H_3) are independent zero mean unit variance circularly symmetric complex Gaussian RVs. The average power of source X and relay X_1 must not exceed powers P and P_1, respectively. It is assumed that the source is not aware of the channel measurements (H_1, H_2, H_3), the relay only knows H_2 and the destination is fully aware of all fading coefficients. Suppose that DF and CF achievable rates are denoted as R_{DF} and R_{CF}.

As the channel is not known to the source, it has to transmit with a fix rate r. If the relay uses DF, or similarly CF, all the time the outage probability is $\Pr(r > R_{\text{DF}})$, or similarly $\Pr(r > R_{\text{CF}})$ for CF case. The following proposition provides a better cooperative strategy for the relay.

Proposition 1. *If the source uses doubly nested lattice code, then the following outage probability is achievable:*

$$\begin{aligned} P_{Out} = \max_{\mathcal{D}_{DF}} & \mathbb{P}(H_2 \in \mathcal{D}_{DF}) \Pr(r > R_{DF} | H_2 \in \mathcal{D}_{DF}) \\ &+ \mathbb{P}(H_2 \notin \mathcal{D}_{DF}) \Pr(r > R_{CF} | H_2 \notin \mathcal{D}_{DF}), \end{aligned}$$

where $\mathcal{D}_{DF} \subseteq \mathbb{C}$ is the decision region for the relay. Moreover the compression noise $\widehat{N}_1$ is chosen only according to H_2.

To provide a rough sketch of the proof, suppose that the source is using doubly nested lattice code. In general only information of H_2 is available at the relay because it does not have any observation channels of the destination. The relay chooses between using DF and CF by only observing H_2. Note that this is possible because the doubly nested lattice code can be used for both choices. This leads naturally to a decision region $\mathcal{D}_{\text{DF}}$ based on which the decision is made. Moreover the relay chooses some constant $\widehat{N}_1$ only based on H_2 and optimize the error probability. It is interesting to see whether the relay can select the proper coding strategy based on H_2. and switches between different cooperative strategies to choose the best one.

In [5], the authors showed that the best decision for the relay is to observe whether it can decode the message based on its channel condition in which case the best choice is to pick DF scheme as cooperative strategy. In other words, it turns out that the optimum decision region $\mathcal{D}_{\text{DF}}$ is given by the set $\mathcal{D}_{\text{DF}}^{\star} =$

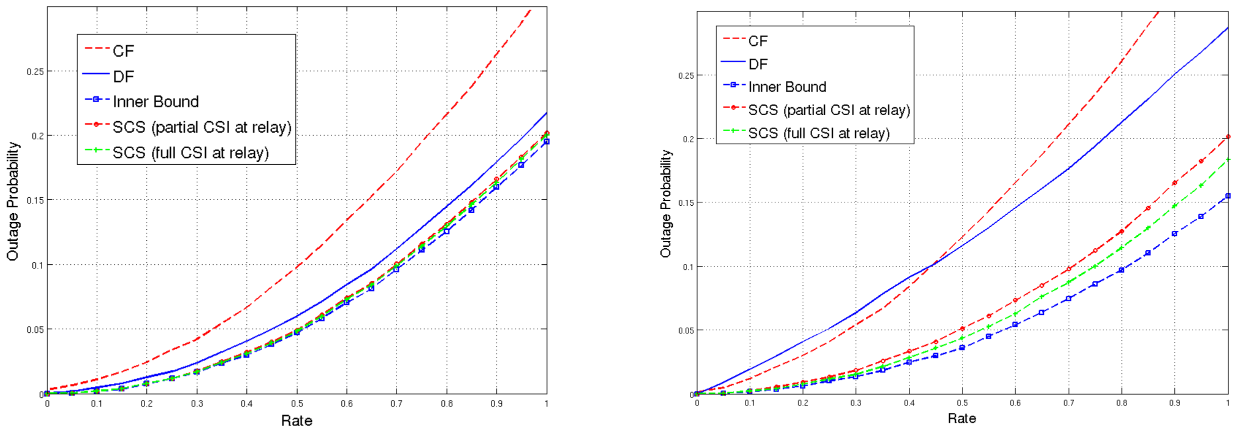

Fig. 3: Outage probability vs. rate for the relay (a) at $d = 0.2$ and (b) at $d = 0.5$

$\left\{H_2 : r \leq \frac{1}{2}\log\left(1+\frac{\overline{\alpha}|H_2|^2 P}{N_1}\right)\right\}$. For the case of partial CSI at the relay, such a decision region outperforms simple DF and CF schemes. Interestingly full CSI is not needed at the relay to take this decision however the full CSI (H_1, H_2, H_3) at the relay improves the error probability only through the choice of the best possible compression noise $\widehat{N}_1$.

A. Numerical Results

In this part, we analyze previous results through numerical results. We assume that $P_1 = P = 1$. We assume all the fading coefficients and noises are of unit variance. All nodes lie on the same line where the source and destination have unit distance and the relay is place at distance d from the source. The standard path loss model is used here with path loss exponent 2. Fig. 3 presents numerical results for $d = 0.2$ and $d = 0.5$. The inner bound on outage probability is obtained using the cutset bound as so called achievable rate. By looking at numerical results, we can see that the selective coding strategy improves upon single cooperative strategies. But if the relay is placed for example very close to the destination or to the source, the channel qualities are ordered for most of the fading realizations according to the relay placement and therefore most of the time a single cooperative strategy performs better. The improvement in this case is not significant compared to single cooperative strategy. In other words, the selective coding strategies improve upon single strategies if the relay placement is such that the fading realizations evenly make one of the strategies superior to the other.

V. Conclusion and Future Works

This work is a first step toward a general selective cooperative strategies for multi-relay networks using structured codes so that relays can switch between DF, CF and Compute-and-Forward. The perquisite of such extension is to establish the obliviousness of lattice codes for single relay channels. In this work, we have proved that doubly nested lattice coding can be used as an oblivious code for the relay channel. Future works include extension of current oblivious lattice coding to Gaussian relay networks.

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