EXPECTED VALUE OF LOGARITHM OF A NON-CENTRAL CHI-SQUARED RANDOM VARIABLE

Abstract. In this note, we derive various closed-form expressions for the expectation of the logarithm of a non-central chi-squared random variable with odd degrees of freedom.

A non-central χ^2 distributed random variable W with m degrees of freedom and non-centrality parameter $\xi = \sum_{i=1}^m \mu_i^2$ is defined as:

$$W = \sum_{i=1}^{m} (X_i + \mu_i)^2, \tag{1}$$

where $\{X_i\}_{i=1}^m \sim \mathcal{N}(0,1)$ and $\{\mu_i\}_{i=1}^m$ are positive constants. The goal is to derive tractable expressions for the expected value of logarithm of W. The first result is given below.

Theorem 1. *Let W be a non-central χ^2 distributed random variable with m degrees of freedom and non-centrality parameter ξ. Then it holds that*

$$\mathbb{E}(\log W) = \mathbb{E}(\psi(Z + m/2)) - \log 2,$$

where Z is a Poisson random variable with parameter $\xi/2$. Equivalently:

$$\mathbb{E}(\psi(Z + m/2)) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\xi^k e^{-\xi/2}}{k!} \psi(k + m/2),$$

and $\psi(\cdot)$ is the digamma function, also known as Euler's psi function:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

Proof. The proof follows closely a similar proof in [1, Lemma 10.1]. The density of W is given by:

$$f_W(x) = \frac{1}{2} \left(\frac{x}{\xi}\right)^{(m-2)/4} e^{-(x+\xi)/2} I_{m/2-1}(\sqrt{\xi x}) \quad x \geq 0,$$

where $I_\nu(\cdot)$ is the modified Bessel function of the first kind of order ν defined as:

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}.$$

Therefore using the series expansion of $I_\nu(\cdot)$, the expected value can be written as:

$$\begin{aligned}
\mathbb{E}(\log W) &= \int_0^\infty \log x \frac{1}{2} \left(\frac{x}{\xi}\right)^{(m-2)/4} e^{-(x+\xi)/2} I_{m/2-1}(\sqrt{\xi x}) \mathrm{d}x \\
&= \int_0^\infty \log x \frac{1}{2} \left(\frac{x}{\xi}\right)^{(m-2)/4} e^{-(x+\xi)/2} \\
&\quad \times \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+m/2)} \left(\frac{\sqrt{\xi x}}{2}\right)^{2k+m/2-1} \mathrm{d}x \\
&= \sum_{k=0}^{\infty} \frac{1}{2^{2k+m/2}} \frac{\xi^k e^{-\xi/2}}{k!\Gamma(k+m/2)} \int_0^\infty \log x e^{-x/2} x^{k+m/2-1} \mathrm{d}x
\end{aligned}$$

The last integral can be simplified using [2, 4.352(1)] as

$$\int_0^\infty \log x e^{-x/2} x^{k+m/2-1} \mathrm{d}x = 2^{k+m/2}\Gamma(k+m/2)(\psi(k+m/2) - \log 2).$$

The final expectation is given by:

$$\begin{aligned}\mathbb{E}(\log W) &= \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\xi^k e^{-\xi/2}}{k!} \psi(k+m/2) - \sum_{k=0}^{\infty} \frac{\log 2}{2^k} \frac{\xi^k e^{-\xi/2}}{k!} \\ &= \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\xi^k e^{-\xi/2}}{k!} \psi(k+m/2) - \log 2.\end{aligned}$$

□

The above theorem expresses the integral formulation of the expected value as an infinite series. The expected value can be approximated arbitrarily well using the series, and it is easier to evaluate in general. A further simplification was obtained for even m. For this case, the digamma function writes as:

$$\psi(k+m/2) = \sum_{j=1}^{k+m/2-1} \frac{1}{j} - \gamma, \tag{2}$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant. This expression was used to derive a simpler expression of the expected value for even m in [3] with the proof in [1]. We provide a similar expression for odd m. The digamma function for odd m writes as:

$$\psi(k+m/2) = -\gamma - 2\log 2 + \sum_{j=1}^{k+(m-1)/2} \frac{2}{2j-1}. \tag{3}$$

The following theorem provides a computationally simpler expression than the theorem above for the expected value of $\log W$.

Theorem 2. *Let W be a non-central χ^2 distributed random variable with odd m degrees of freedom and non-centrality parameter ξ. Then it holds that*

$$\mathbb{E}(\log W) = C_0 + \frac{1}{C_m}\left(\frac{1}{\sqrt{\pi}} h_1(\xi/2) - \tilde{h}_m(\xi/2)\right),$$

where:

$$\begin{aligned} C_0 &= -\gamma - 3\log 2, \quad C_m = \frac{(-1)^{[m/2]}}{\Gamma(m/2)}, \\ h_1(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \log(x+\sqrt{2t})^2 e^{-x^2/2} \mathrm{d}x + \gamma + 2\log(2), \\ \tilde{h}_m(t) &= \sum_{j=0}^{[m/2]-1} \frac{t^{j+1}(-1)^j}{(j+1)\Gamma(j+3/2)}. \end{aligned}$$

Proof. For m odd, we have:

$$\mathbb{E}(\psi(Z+m/2)) - \log 2 = -\gamma - 3\log 2 + \mathbb{E}\left(\sum_{j=1}^{Z+(m-1)/2} \frac{2}{2j-1}\right).$$

Define the function $h_m(t)$ as:

$$h_m(t) = \mathbb{E}\left(\sum_{j=1}^{Z+(m-1)/2} \frac{2}{2j-1}\right) = \sum_{k=0}^{\infty} \frac{e^{-t} t^k}{k!} \left(\sum_{j=1}^{k+(m-1)/2} \frac{2}{2j-1}\right),$$

and see that:

$$h_m'(t) = \sum_{k=0}^{\infty} \frac{e^{-t} t^k}{k!} \frac{1}{k+m/2}.$$

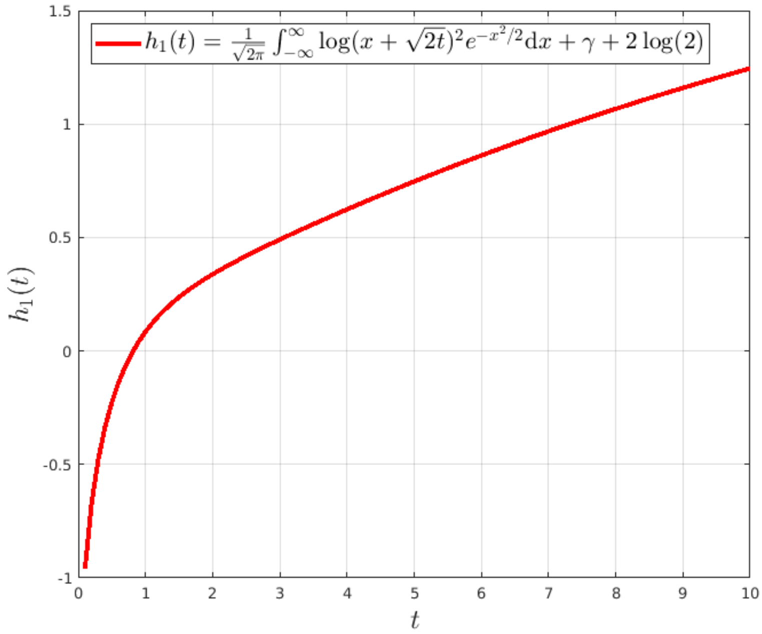

FIGURE 1. The function $h_1(t)$

We find another representation of $h'_m(t)$ that contains only finite sums and provides a more concise version of $h_m(t)$. The following lemma, a generalization of a familiar combinatorial equality, is used for the proof.

Lemma 1. *Let $x \in \mathbb{R}^+$ and $k \in \mathbb{N}$ such that $k > x$. We have:*

$$\sum_{j=0}^{[x]-1} \frac{(-1)^j}{\Gamma(j+\{x\}+1)(k-j)!} = -\frac{x(-1)^{[x]}}{(k+\{x\})(k-[x])!\Gamma(x+1)} + \frac{\{x\}}{(k+\{x\})\Gamma(\{x\}+1)k!}.$$

The previous lemma yields that:

$$\sum_{j=0}^{[m/2]-1} \frac{(-1)^j}{\Gamma(j+3/2)(k-j)!} = -\frac{m(-1)^{[m/2]}}{2(k+1/2)(k-[m/2])!\Gamma(m/2+1)} + \frac{1}{2(k+1/2)\Gamma(3/2)k!}. \tag{4}$$

Consider the following expression of $h'_m(t)$:

$$h'_m(t) = \sum_{k=0}^{\infty} \frac{e^{-t}t^k}{k!} \frac{1}{k+m/2} = e^{-t} \sum_{k=[m/2]}^{\infty} \frac{t^k}{(k-[m/2])!} \frac{1}{k+1/2},$$

Using (4) with the above equality, we obtain:

$$\begin{aligned} &\frac{m(-1)^{[m/2]}}{2\Gamma(m/2+1)} \sum_{k=[m/2]}^{\infty} \frac{e^{-t}t^k}{(k+1/2)(k-[m/2])!} \\ &\qquad = \sum_{k=[m/2]}^{\infty} \frac{e^{-t}t^k}{2(k+1/2)\Gamma(3/2)k!} - \sum_{k=[m/2]}^{\infty} \sum_{j=0}^{[m/2]-1} \frac{e^{-t}t^k(-1)^j}{\Gamma(j+3/2)(k-j)!}. \end{aligned}$$

Let us focus on each of these terms. Consider the first term:

$$\begin{aligned} \sum_{k=[m/2]}^{\infty} \frac{e^{-t}t^k}{2(k+1/2)\Gamma(3/2)k!} &= \frac{1}{\sqrt{\pi}} \sum_{k=[m/2]}^{\infty} \frac{e^{-t}t^k}{(k+1/2)k!} \\ &= \frac{1}{\sqrt{\pi}} h'_1(t) - \frac{1}{\sqrt{\pi}} \sum_{k=0}^{[m/2]-1} \frac{e^{-t}t^k}{(k+1/2)k!}. \end{aligned}$$

Note that:

$$h_1(t) = \mathbb{E}(\log W') + \gamma + 2\log 2, \tag{5}$$

where W' is equal to $(X+\sqrt{2t})^2$ for X as a standard normal distributed random variable:

$$h_1(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\log(x+\sqrt{2t})^2 e^{-x^2/2}\mathrm{d}x + \gamma + 2\log(2).$$

The second term can be characterized as:

$$\begin{aligned}
&\sum_{k=[m/2]}^{\infty}\sum_{j=0}^{[m/2]-1}\frac{e^{-t}t^k(-1)^j}{\Gamma(j+3/2)(k-j)!}\\
&=\sum_{i=0}^{\infty}\sum_{j=0}^{[m/2]-1}\frac{e^{-t}t^{i+j}(-1)^j}{\Gamma(j+3/2)i!} \quad -\sum_{k=0}^{[m/2]-1}\sum_{j=0}^{k}\frac{e^{-t}t^k(-1)^j}{\Gamma(j+3/2)(k-j)!},
\end{aligned}$$

but then:

$$\begin{aligned}
\sum_{i=0}^{\infty}\sum_{j=0}^{[m/2]-1}\frac{e^{-t}t^{i+j}(-1)^j}{\Gamma(j+3/2)i!} &= \sum_{i=0}^{\infty}\frac{e^{-t}t^i}{i!}\sum_{j=0}^{[m/2]-1}\frac{t^j(-1)^j}{\Gamma(j+3/2)}\\
&= \sum_{j=0}^{[m/2]-1}\frac{t^j(-1)^j}{\Gamma(j+3/2)}.
\end{aligned}$$

Assuming $C_m = \frac{m(-1)^{[m/2]}}{2\Gamma(m/2+1)} = \frac{(-1)^{[m/2]}}{\Gamma(m/2)}$, we have:

$$\begin{aligned}
C_m h_m'(t) &= \frac{1}{\sqrt{\pi}}h_1'(t) - \frac{1}{\sqrt{\pi}}\sum_{k=0}^{[m/2]-1}\frac{e^{-t}t^k}{(k+1/2)k!}\\
&-\sum_{j=0}^{[m/2]-1}\frac{t^j(-1)^j}{\Gamma(j+3/2)} + \sum_{k=0}^{[m/2]-1}\sum_{j=0}^{k}\frac{e^{-t}t^k(-1)^j}{\Gamma(j+3/2)(k-j)!}.
\end{aligned}$$

It can be seen that:

$$\frac{1}{\sqrt{\pi}}\frac{1}{(k+1/2)k!} = \sum_{j=0}^{k}\frac{(-1)^j}{\Gamma(j+3/2)(k-j)!},$$

which implies that:

$$\frac{1}{\sqrt{\pi}}\sum_{k=0}^{[m/2]-1}\frac{e^{-t}t^k}{(k+1/2)k!} = \sum_{k=0}^{[m/2]-1}\sum_{j=0}^{k}\frac{e^{-t}t^k(-1)^j}{\Gamma(j+3/2)(k-j)!},$$

hence:

$$C_m h_m'(t) = \frac{1}{\sqrt{\pi}}h_1'(t) - \sum_{j=0}^{[m/2]-1}\frac{t^j(-1)^j}{\Gamma(j+3/2)}. \tag{6}$$

A simple integration with adjustment of the respective constant shows that:

$$C_m h_m(t) = \frac{1}{\sqrt{\pi}}h_1(t) - \sum_{j=0}^{[m/2]-1}\frac{t^{j+1}(-1)^j}{(j+1)\Gamma(j+3/2)},$$

with $h_1(t)$ given in (5). The function $h_1(t)$ is given in Figure 1. The proof follows from the above result. □

References

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