Generalised water-filling: costly power optimally allocated to sub-carriers under a general concave performance function

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Executive Overview

- We generalise the standard “water-filling” scenario: we allocate a power budget to a number of subchannels with a price on power, and sub-channel performance given by a general concave function of the power allocated to the sub-carrier.

Philosophy: market as a paradigm for algorithms

- A complex technological system can be “efficiently” managed as an “economy”
- The system can be viewed as integrated by many “agents”
- Agents may correspond to actual human beings, or may be machines, or processes
- The system administrator sets up some relatively simple rules for resource use and behaviour (prices, auctions, rewards, punishments, etc)
- Each agent behaves and utilises resources as an economic entity seeking to maximise its “preferences” while obeying the rules and budget constraints (energy, power, bandwidth, etc).
- If the rules are “right”, the complex system produces “efficient” results

Problem statement

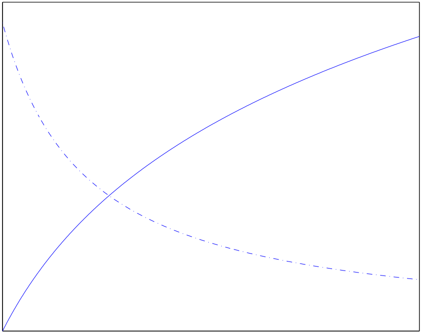
- One terminal, M sub-channels, total power constraint
- sub-channel state represented by h_m : power gain over noise
  (for convenience $h_1 \geq \cdots \geq h_M > 0$)
- total power constraint: P
- 1 unit of power costs c_0 (\$/Watt)
- 1 bit is worth to the terminal b_0 (\$/bit)
- General $f_0 : \Re_+ \to \Re_+$ yields performance (bits) from sub-channel signal-to-noise ratio (SNR) $x_m = h_m p_m$
- Terminal maximises benefit minus cost ("surplus"):

$$\max_{x_1,\cdots,x_M} b_0 \sum_{m=1}^{M} f_0(x_m) - c_0 \sum_{m=1}^{M} \frac{x_m}{h_m}$$

$$\text{s.t. } \sum_{m=1}^{M} \frac{x_m}{h_m} \leq P \tag{1}$$

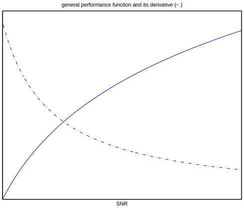
$$x_m \geq 0$$

General concave performance function of SNR



- $f_0 : \Re_+ \to \Re_+$
- strictly increasing
- concave (f_0' is strictly decreasing)
- $f_0(0) = 0$
- $f_0'(0) < \infty$
- $\lim_{t\to\infty} f_0'(t) = 0$
- EXAMPLE: $f_S(t) := \ln(1 + t)$

1st step: Normalisation



- objective function can be re-written as

$$b_0 f_0'(0) \left(\sum_{m=1}^{M} \frac{f_0(x_m)}{f_0'(0)} - \frac{c_0}{b_0 f_0'(0)} \sum_{m=1}^{M} \frac{x_m}{h_m} \right)$$

- Let $f(t) := f_0(t)/f_0'(0)$ ($f'(0) = 1$)
- with $c := c_0/(b_0 f_0'(0))$ problem can be re-stated as:

$$\max_{x_1,\cdots,x_M} \sum_{m=1}^{M} f(x_m) - c \sum_{m=1}^{M} \frac{x_m}{h_m}$$
$$\text{s.t. } \sum_{m=1}^{M} \frac{x_m}{h_m} \leq P \tag{2}$$
$$x_m \geq 0$$

First-order necessary optimising conditions (FONOC)

Fact

If $(x_1^*, \cdots, x_M^*)$ is a (local) optimiser corresponding to Problem (1), then there are non-negative real numbers λ, $\mu_1, \cdots .\mu_M$ such that

$$h_m f'(x_m^*) = c + \lambda - \mu_m \quad \forall m \tag{3}$$

$$\lambda \left(\sum_{m=1}^{M} \frac{x_m^*}{h_m} - P \right) = 0 \tag{4}$$

$$\mu_m \frac{x_m^*}{h_m} = 0 \quad \forall m \tag{5}$$

First result: the normalised power cost c defines a threshold for channel usability; if $h_m \leq c$ then the sub-channel is useless; i.e., if $(x_1^*, \cdots, x_M^*)$ is a (local) optimiser then $h_m < c \implies x_m^* = 0$

Two solutions to FONOC

Two cases of interest:

- Power is “plentiful” (power-constraint multiplier $\lambda = 0$)
- Power is “scarce” ($\lambda > 0$)

Plentiful power solution ($\lambda = 0$)

- each usable sub-channel is allocated its individual maximiser ("greedy" solution); that is, the choice that maximises benefit minus cost on a channel-by-channel basis.
- The "greedy" optimiser is defined by $h_m f'(h_m p_m) = c$
- has a "water-filling" interpretation in logarithmic ("dB") scale: $\log(h_m) + \log(f'(h_m p_m)) = \log(c)$.
- for $f(t) = \ln(1+t)$ it leads to water-filling in the natural scale over the *usable* sub-channels:

$$p_m + 1/h_m = 1/c$$

Scarce power solution

- "greedy solution" may exceed the total power constraint
- in this case , λ becomes non-zero
- system behaves as if the power price was $c+\lambda$ ("effective" price):
$$h_m f'(h_m p_m) = c+\lambda$$
- $c+\lambda^*$ makes total power consumption ("demand") equal to the total available "supply" (i.e., the "market clearance" price).
- To find λ^* set total allocated power equal to constraint (some iterations may be needed).
- An initially usable sub-channel ($h_m > c$) may become unusable: $h_m \leq c+\lambda^*$
- For logarithmic f,
$$c+\lambda^* = \frac{m_0 - 1}{P + \sum_{m=1}^{m_0-1} \frac{1}{h_m}}$$

Summary

- The sub-channel power allocation problem and its "water-filling" solution are well-known, when
  - the performance ("capacity") function is logarithmic and
  - power is constrained but costless.
- We have generalised this problem by considering
  - a general concave performance function and
  - a per-Watt price.
- The solution retains the general water-filling structure, but the costly power does change matters:
  - the normalised power price translates to a channel gain: if a noise-normalised sub-channel gain $h_m < c$ the sub-channel is not usable: marginal benefit of buying a unit of power is less than its cost (all sub-channels be discarded)
  - Basic equation: $h_m f'(h_m p_m) = c + \lambda$
- Our analysis can be applied:
  - directly as a price-driven power-allocation, for time-division OFDM
  - for OFDMA, if combined with a good sub-channel allocation scheme (see our other paper!)